November Problem of the Month

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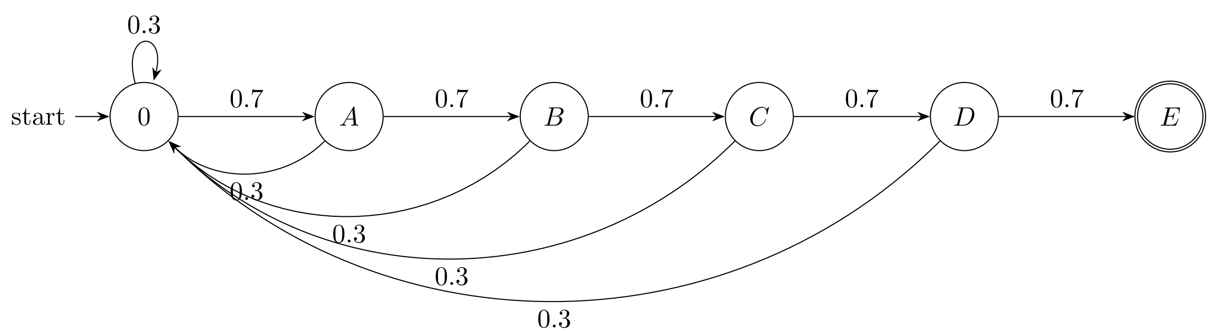
Problem. Frank Ilett, a dedicated Manchester United (MU) fan, promised not to get a haircut until his team wins five consecutive games. MU has recently improved and now has a 70% (probability = 0.7) of winning any individual game. We want to answer the following question: on average how many games will Frank need to watch before MU wins five games in a row and he can get his haircut?

We will use a Markov chain to represent the states, moving towards E as we see more wins.



"Unrolling" this Markov chain, we get

$$
\begin{gathered}
\mathrm{P}(0) = 1 + 0.7\mathrm{P}(\mathrm{A}) + 0.3\mathrm{P}(0) \\
\mathrm{P}(\mathrm{A}) = 1 + 0.7\mathrm{P}(\mathrm{B}) + 0.3\mathrm{P}(0) \\
\mathrm{P}(\mathrm{B}) = 1 + 0.7\mathrm{P}(\mathrm{C}) + 0.3\mathrm{P}(0) \\
\mathrm{P}(\mathrm{C}) = 1 + 0.7\mathrm{P}(\mathrm{D}) + 0.3\mathrm{P}(0) \\
\mathrm{P}(\mathrm{D}) = 1 + 0.7\mathrm{P}(\mathrm{E}) + 0.3\mathrm{P}(0) \\
\mathrm{P}(\mathrm{E}) = 0
\end{gathered}
$$

Solving, we get

$$\begin{gathered}\mathrm{P(E)} = 0 \\ \mathrm{P(D)} \approx 5.949 \\ P(C) \approx 10.115 \\ P(B) \approx 13.030 \\ P(A) \approx 15.071 \\ P(0) \approx 16.499\end{gathered}$$

Therefore, MU would have to play roughly 17 games to win 5 in a row.

```
import random
def simulate_games_until_consecutive_wins(win_probability):
    total_games_played = 0
    consecutive_wins = 0
    required_consecutive_wins = 5

    while consecutive_wins < required_consecutive_wins:
        total_games_played += 1
        if random.random() <= win_probability:
            consecutive_wins += 1
        else:
            consecutive_wins = 0
    return total_games_played


win_probability = 0.7  # Probability of MU winning a single game
num_trials = 1000000    # Number of simulations to run

results = []

for _ in range(num_trials):
    games_played = simulate_games_until_consecutive_wins(win_probability)
    results.append(games_played)


average_games = sum(results) / len(results)
print(f"Average number of games to achieve 5 consecutive wins: {average_games:.2f}")
```