PROBLEM OF THE MONTH

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Problem. Let $p_1, p_2, \ldots, p_m$ be distinct prime numbers. Show that

$$\log(p_1), \log(p_2), \ldots, \log(p_m) \in \mathbb{R}$$

are linearly independent over $\mathbb{Q}$. (Here, $\log(x)$ denotes the natural logarithm.) Use this fact to show that $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{Q}$.

The idea below is due to Dung Nguyen.

Proof. Suppose

$$q_1 \log p_1 + \cdots + q_m \log p_m = 0$$

for some $q_1, \ldots, q_m \in \mathbb{Q}$. We will show that

$$q_1 = \cdots = q_m = 0.$$

Choose a positive integer N such that $Nq_i \in \mathbb{Z}$ for every i. Multiplying the equation by N, we get

$$a_1 \log p_1 + \cdots + a_m \log p_m = 0,$$

where $a_i = Nq_i \in \mathbb{Z}$.

Using the properties of logarithms,

$$\log\left(p_1^{a_1} \cdots p_m^{a_m}\right) = 0.$$

Therefore,

$$p_1^{a_1} \cdots p_m^{a_m} = 1.$$

Since $p_1, \ldots, p_m$ are distinct primes, the uniqueness of prime factorization implies that

$$a_1 = \cdots = a_m = 0.$$

Hence

$$q_1 = \cdots = q_m = 0.$$

Thus

$$\log p_1, \log p_2, \ldots, \log p_m$$

are linearly independent over $\mathbb{Q}$.

Now there are infinitely many prime numbers. Therefore the set

$$\{\log p : p \text{ is prime}\}$$

is an infinite linearly independent subset of $\mathbb{R}$ over $\mathbb{Q}$. Since a finite-dimensional vector space cannot contain an infinite linearly independent set, it follows that $\dim_{\mathbb{Q}} \mathbb{R} = \infty$. $\square$

Here is another solution due to Tung Nguyen for the fact that $\dim_{\mathbb{Q}} \mathbb{R} = \infty$.

Proof. Suppose, for contradiction, that $\mathbb{R}$ is finite-dimensional over $\mathbb{Q}$. Then there exists a basis

$$v_1, \ldots, v_n$$

of $\mathbb{R}$ over $\mathbb{Q}$. Thus every $x \in \mathbb{R}$ can be written uniquely as

$$x = q_1 v_1 + \cdots + q_n v_n, \qquad q_1, \ldots, q_n \in \mathbb{Q}.$$

Since $\mathbb{Q}$ is countable, $\mathbb{Q}^n$ is also countable. Therefore there are only countably many possible linear combinations

$$q_1 v_1 + \cdots + q_n v_n.$$

Hence $\mathbb{R}$ would be countable.

But $\mathbb{R}$ is uncountable, a contradiction. $\square$