Perfect state transfer on gcd-graphs over a finite Frobenius ring

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What is a graph?

A (undirected) graph is an ordered pair $G = (V, E)$ where

- V is a finite set whose elements are called vertices,
- E is a set of paired vertices.

Suppose the vertex set of G is $\{v_1, v_2, \ldots, v_n\}$. A convenient way to represent G is to use its adjacency matrix $A = A_G = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{else.} \end{cases}$$

With this presentation, we can then use tools from matrix theory, representation theory, and number theory to study the structure of G.

An Erdős–Rényi random graph

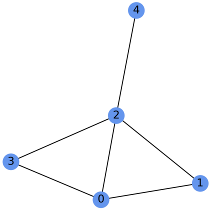

Figure 1: A random graph on $n = 5$ nodes

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The adjacency matrix of this graph.

- The spectrum of G, denoted by $\mathrm{Spec}(G)$, is the set of all eigenvalues of its adjacency matrix A. Equivalently, it is the set of all roots of the characteristic polynomial $p_A(t)$ of A where

$$p_A(t) = \det(tI_n - A).$$

- Let K be a subfield of $\mathbb{C}$. A graph is called K-rational if $\lambda \in \mathcal{O}_K$ for each $\lambda \in \mathrm{Spec}(G)$ where $\mathcal{O}_K$ is the ring of integers in K.
- A $\mathbb{Q}$-rational graph is often called an integral graph.

Perfect state transfer on graphs

Definition

Let $F(t)$ be the continuous-time quantum walk associated with G; namely $F(t) = \exp(\mathrm{i}A_G t)$. There is perfect state transfer (PST) in graph G if there are distinct vertices a and b and a positive real number t such that $|F(t)_{ab}| = 1$.

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$$F(t) = \cos(t)I + \mathrm{i}\sin(t)A = \begin{bmatrix} \cos(t) & \mathrm{i}\sin(t) \\ \mathrm{i}\sin(t) & \cos(t) \end{bmatrix}.$$

and hence

$$F\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

This shows that there is PST between u and v at $t = \frac{\pi}{2}$.

Cayley graphs over a finite commutative ring

In general, the classification of PST on a graph is a difficult problem. However, for certain arithmetic graphs, this problem is more tractable.

Definition

Let R be a finite commutative group and S is a subset of R. The Cayley graph $G = \Gamma(R, S)$ is the graph with the following data

- $V(G) = R$.
- $a, b \in V(G)$ *are adjacent if* $a - b \in S$.

In practice, the definition of S often involves the multiplicative structure of R.

Integral Cayley graphs

Theorem (Godsil)

Suppose that there is PST on G.

1. *G is K-rational where K is either $\mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$.*
2. *If G is regular, then it is $\mathbb{Q}$-rational.*

- We can classify all integral Cayley graphs defined over R (works of Godsil-Spiga, So, Nguyen-Tân). More on this later.
- The classification of Cayley graphs with PST seems to be a much harder problem.

The Circulant Diagonalization Theorem

Let G be a Cayley graph defined over $R = \mathbb{Z}/3$. The adjacency matrix of G is a 3×3 matrix of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{pmatrix}.$$

Let ω_3 be 3-root of unity; namely $\omega_3^3 = 1$. Then we have

$$C \begin{pmatrix} 1 \\ \omega_3 \\ \omega_3^2 \end{pmatrix} = \begin{pmatrix} c_0 + c_1\omega_3 + c_2\omega_3^2 \\ c_2 + c_0\omega_3 + c_1\omega_3^2 \\ c_1 + c_2\omega_3 + c_0\omega_3^2 \end{pmatrix} = \begin{pmatrix} (c_0 + c_1\omega_3 + c_2\omega_3^2)1 \\ (c_0 + c_1\omega_3 + c_2\omega_3^2)\omega_3 \\ (c_0 + c_1\omega_3 + c_2\omega_3^2)\omega_3^2 \end{pmatrix}.$$

We see that $(1, \omega_3, \omega_3^2)^t$ is an eigenvector of C associated with the eigenvalue $c_0 + c_1\omega_3 + c_2\omega_3^2$.

The Circulant Diagonalization Theorem

More generally we have the following theorem.

Theorem (Circulant Diagonalization Theorem)

Let $G = \Gamma(R, S)$ be a Cayley graph. Then, the spectrum of G is precisely the multiset

$$\{\lambda_\chi = \sum_{s \in S} \chi(s)\}_{\chi \in \widehat{R}},$$

here $\widehat{R} = Hom(R, \mathbb{C}^\times)$ is the dual group of R considered as an abelian group.

The spectrum of G is precisely the Discrete Fourier Transform of the indicator vector of S.

Finite Frobenius rings

Let n be the characteristic of R and let $\zeta_n := e^{\frac{2\pi i}{n}}$ be a primitive root of unity.

Definition

A finite commutative ring R is called Frobenius if there exists a $\mathbb{Z}/n$-functional $\psi : R \to \mathbb{Z}/n$ such that $\ker(\psi)$ does not contain any non-zero ideal in R.

- For each $r \in R$, define $\chi_r \in \widehat{R}$ by the rule

$$\chi_r(s) = \zeta_n^{\psi(rs)}.$$

- The fact that $\ker(\psi)$ does not contain any non-zero ideal in R implies that the map $R \to \widehat{R}$ defined by $r \mapsto \chi_r$ is an isomorphism. In other words, R is canonically self-dual.

Finite Frobenius rings

Some examples of finite Frobenius rings.

- $R = \prod_n \mathbb{Z}/n$. Consequently, each finite abelian group is isomorphic to a $(R, +)$ where R is a finite Frobenius ring.
- R is a finite quotient of $\mathcal{O}_K$ where K is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$.
- If R is Frobenius and H is an abelian group then $R[H]$ is also Frobenius.
- Every finite commutative ring is a quotient of a finite Frobenius ring.

- Let $G = \Gamma(R, S)$ be a Cayley graph defined over R.
- For each $r \in R$, we define

$$\vec{v}_r = \frac{1}{\sqrt{|R|}}[\zeta_n^{\psi(rs)}]^T_{s \in R} \in \mathbb{C}^{|R|}, \lambda_r = \sum_{s \in S} \zeta_n^{\psi(rs)}.$$

  Then v_r is a normalized eigenvector of A_G with λ_r being the corresponding eigenvalue.
- Let $V = [v_r]_{r \in R} \in \mathbb{C}^{|R| \times |R|}$, $D = \text{diag}([\lambda_r]_{r \in R})$. Then we can write

$$A_G = VDV^* = \sum_{r \in R} \lambda_r \vec{v}_r \vec{v}_r^*,$$

- Therefore, we have

$$F(t) = \sum_{r \in R} e^{\mathrm{i}\lambda_r t} \vec{v}_r \vec{v}_r^*.$$

- Let $s_1, s_2 \in R$. Then

$$F(t)_{s_1,s_2} = \frac{1}{|R|}\sum_{r\in R} e^{\mathrm{i}\lambda_r t}\zeta_n^{\psi((s_1-s_2)r)}$$
$$= \frac{1}{|R|}\sum_{r\in R} e^{2\pi\mathrm{i}\left(\lambda_r \frac{t}{2\pi}+\frac{\psi((s_1-s_2)r)}{n}\right)}.$$

- By the triangle inequality, $|F(t)_{s_1,s_2}| = 1$ if and only if $\lambda_r \frac{t}{2\pi} + \frac{\psi(s_1-s_2)r}{n}$ are the same modulo 1 for all $r \in R$.
- By symmetry, there exists perfect state transfer between s_1 and s_2 if and only if there exists perfect state transfer between 0 and $s_2 - s_1$.

In summary, we have the following.

Theorem (Nguyen-Tân, Bašić-Petković-Stevanović)

Let $G = \Gamma(R, S)$ be a Cayley graph defined over a finite Frobenius ring. There exists perfect state transfer from 0 to s at time t if and only if for all $r_1, r_2 \in R$

$$(\lambda_{r_1} - \lambda_{r_2})\frac{t}{2\pi} + \frac{\psi(s(r_1 - r_2))}{n} \in \mathbb{Z}\,.$$

Corollary

Let Δ be the abelian group generated by $r_1 - r_2$ where r_1 and r_2 are elements of R such that $\lambda_{r_1} = \lambda_{r_2}$. Then $\psi(sd) = 0$ for all $d \in \Delta$. In particular, if $\Delta = R$ then there is no PST on $\Gamma(R, S)$.

Gcd-graphs over a Frobenius ring

Definition

A Cayley graph $\Gamma(R, S)$ is called a gcd-graph if S is stable under the action of $R^\times$.

- We can show that S is stable under the action of $R^\times$ if and only if there exists a subset $D = \{x_1, x_2, \ldots, x_k\}$ of non-associate elements in R with the property that: for each $s \in R$, $s \in S$ if and only if $sR = x_i R$ for some $x_i \in D$.
- Gcd-graphs over a finite quotient of $\mathbb{Z}$ were introduced by Klotz-Sander. They described the spectra of these graphs using the theory of Ramanujan sums $c_m(n)$ where

$$c_n(m) = \sum_{\substack{1 \le j \le n \\ \gcd(j,n)=1}} \zeta_n^{mj}.$$

- Various generalizations to more general rings: Thongsomnuk-Meemark (for a principal ideal ring) and Nguyen-Tân (for a general commutative ring).

An example

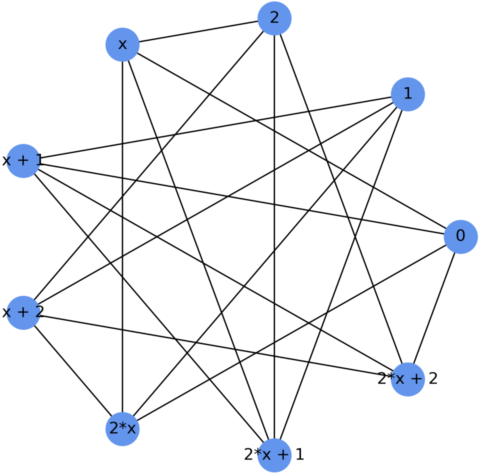

Figure 2: The gcd-graph $G_f(D)$ with $f = x(x+1) \in \mathbb{F}_3[x]$ and $D = \{x, x+1\}$

- In practice, it is often the case that S is only stable under the action of a proper subgroup U of $R^{\times}$. We call these graphs U-unitary Cayley graphs.
- In this case, the spectra of these graphs can be described by generalized Ramnujan sums, which can be further explained by the supercharacter theory on R associated with U (work of Nguyen-Tân).
- A Cayley graph $\Gamma(R, S)$ is integral if and only if S is U-unitary where $U = (\mathbb{Z}/n)^{\times}$.

Perfect state transfer on gcd- graphs

Question

Can we classify all gcd-graphs on R that have PST?

When $R = \mathbb{Z}/n$, many results are known (due to works of Godsil, Bašić-Petković-Stevanović, and others)

- PST can only exist between 0 and $n/2$. In particular, n must be even.
- When $S = (\mathbb{Z}/n)^{\times}$, PST exists only for $n = 2, 4$.

Perfect state transfer on gcd- graphs

Theorem (Nguyen-Tân)

Let R be a finite Frobenius ring. Suppose that R has the following Artin-Wedderburn decomposition: $R = (\prod_{i=1}^{d} S_i) \times R_2$. Here, $(S_i, \mathfrak{m}_i)$ represents all local factors of R whose residue fields are $\mathbb{F}_2$. For each $1 \leq i \leq d$, let e_i be the unique minimal element of S_i.

1. *If there exists PST between 0 and some $s \in R$, then s must be of the form $(a_1, a_2, \ldots, a_d, 0)$, where each a_i is 0 or e_i. In particular, if R is a local ring, then $s = e$, where e is the unique minimal element of S.*
2. *Suppose that $(R, \mathfrak{m})$ is a principal ideal local ring with a generator α and residue field $\mathbb{F}_2$. Let n be the smallest positive integer such that $\alpha^n = 0$. Then, the gcd-graph $\Gamma(R, S)$ has PST if and only if $|S \cap \{\alpha^{n-1}, \alpha^{n-2}\}| = 1$.*

Perfect state transfer on gcd- graphs

Let us consider the case $S = R^\times$. In this case, we have the following theorem.

Theorem (Thongsomnuk-Meemark, Nguyen-Tân)

There exists PST on $\Gamma(R, R^\times)$ *if and only if* $R = S_1 \times \prod_{i=1}^{r} \mathbb{F}_{q_i}$ *where*

1. S_1 *is a product of local rings* S' *where each* $S' \in \{\mathbb{F}_2, \mathbb{Z}/4, \mathbb{F}_2[x]/x^2\}$.
2. $q_i \equiv 1 \pmod 4$ *for all* i.

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