ISOMORPHIC GCD-GRAPHS OVER POLYNOMIAL RINGS

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Dedicated to Professor Ki-Bong Nam on the occasion of his 70th birthday

ABSTRACT. Gcd-graphs over the ring of integers modulo n are a simple and elegant class of integral graphs. The study of these graphs connects multiple areas of mathematics, including graph theory, number theory, and ring theory. In a recent work, inspired by the analogy between number fields and function fields, we define and study gcd-graphs over polynomial rings with coefficients in finite fields. We discover that, in both cases, gcd-graphs share many similar and analogous properties. In this article, we extend this line of research further. Among other topics, we explore an analog of a conjecture of So and a weaker version of Sander-Sander, concerning the conditions under which two gcd-graphs are isomorphic or isospectral. We also provide several constructions showing that, unlike the case over $\mathbb{Z}$, it is not uncommon for two gcd-graphs over polynomial rings to be isomorphic.

1. INTRODUCTION

Unitary graphs and their natural generalizations, gcd-graphs, over $\mathbb{Z}$ are first formally introduced by the work of Klotz and Sander (see [6]). Since this work, the literature has seen an explosion of research around the topics exploring many further fundamental properties of these graphs including their connectedness, bipartiteness, perfectness, clique and independence numbers, spectral properties, and much more. We refer the readers to [1, 2, 4, 5] and the references therein for further discussions around this line of research.

We first recall the definition of a gcd-graph.

Definition 1.1. Let A be a PID and $n \in A$ which is not a unit. Suppose further that the ring A/n is finite. Let $\mathrm{Div}(n)$ be the set of proper divisors of n (defined up to an associate) and $D \subset \mathrm{Div}(n)$. The gcd-graph $G_n(D)$ is the graph equipped with the following data

(1) The vertex set of $G_n(D)$ is A/n.
(2) Two vertices $a, b \in G_n$ are adjacent if and only if $\gcd(a-b, n) \in D$.

In other words, $G_n(D)$ is the Cayley graph on A/n with the generating set

$$S_D = \{h \in A/n \mid \gcd(h, n) \in D.\}$$

The case $A = \mathbb{Z}$ is discussed in [6] and the case $A = \mathbb{F}_q[x]$ is the main topic of [8]. By their own nature, we can see that the study of these gcd-graphs graphs bridges several branches

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of mathematics including graph theory, number theory, commutative algebra, and character theory for finite groups. For example when $A = \mathbb{Z}$, using the the theory of Ramanujan sums, the authors of [6] show that gcd-graphs are integral; meaning all of their eigenvalues are integers. In [13], So shows that the converse is true as well: a $\mathbb{Z}/n$-circulant graph is integral if and only if it is a gcd-graph. So also poses a conjecture about whether a gcd-graph $G_n(D)$ determined D and n (see [13, Conjecture 7.3]). While this conjecture is still open, some progress has been made. For example, Sander-Sander in [11] asks that given n whether D is determined by the spectral vector $(\sum_{d \in D} c(\ell, \frac{n}{d}))_{\ell=1}^{n}$ where $c(\ell, \frac{n}{d})$ is the Ramanujan sum (see [6, Section 4] for the definition). We remark that the components of this spectral vector are exactly the eigenvalues of $G_n(D)$ counted with multiplicity. In the same paper, Sander and Sander prove their own conjecture. In [12], Schlage-Puchta gives a new and shorter proof for the weak conjecture of Sander-Sander using a determinant involving Ramanujan sums.

Given what is already known in the case $A = \mathbb{Z}$, one may ask whether the conjecture of So and the weaker version of Sander-Sander hold in the case $A = \mathbb{F}_q[x]$. As we will see in this article, the answer is NO for the first question (though it fails for a good reason, see Remark 4.14) and YES for the second question. In fact, for the first question, we will provide various constructions of isomorphic gcd-graphs for different f and D. One particular reason for this stark difference between the number case and the function field case is that over function fields, we can find $f \neq g$ such that $\mathbb{F}_q[x]/f \cong \mathbb{F}_q[x]/g$ whereas in the number field case, this is impossible. For the second question, we will show that the approach of [12] can be adapted naturally in the function field setting leading to a proof of the weak conjecture of Sander-Sander in this case.

1.1. **Outline.** In Section 2, we prove the analog of the weaker conjecture of Sander-Sander in the function fields setting. More precisely, we show that for a fixed $f \in \mathbb{F}_q[x]$, D is determined by a spectral vector describing all eigenvalues of $G_f(D)$. We achieve this by studying a matrix involving Ramanujan sums whose counterpart over $\mathbb{Z}$ was introduced in [12]. In Section 3, we study gcd-graphs $G_f(D)$ where f is a prime power. Here, we show that many of the results concerning the spectrum of $G_f(D)$, as described in [10, 11], have analogs in the function field setting. Furthermore, we explore several fundamental graph-theoretic properties of $G_f(D)$ such as their bipartiteness, perfectness, clique, and independent numbers, which, to the best of our knowledge, have not yet been addressed in the literature—even for gcd-graphs over $\mathbb{Z}$. Section 4 studies isomorphism between gcd-graphs. This is where we will see differences between gcd-graphs over $\mathbb{Z}$ and gcd-graphs over $\mathbb{F}_q[x]$. By analyzing experimental data via the Python library Networkx, we provide several constructions of isomorphic gcd-graphs.

1.2. **Code.** Many insights in this paper are gained through an extensive analysis of experimental data. The code that we wrote to generate data and do experiments on them can be found at [9]. Additionally, we have also verified all statements in this work with various concrete and computable examples.

2. SPECTRAL VECTORS DETERMINE D

Let q be a prime power, and $f \in \mathbb{F}_q[x]$ a monic polynomial. As we explain in [8, Section 6], there is a direct analogy between the character theory of $\mathbb{Z}/n$ and that of $\mathbb{F}_q[x]/f$. More specifically, while the character theory of $\mathbb{Z}/n$ can be fully described once we fix a primitive n-th root of unity, the character theory of $\mathbb{F}_q[x]/f$ is similarly determined by a fixed non-degenerate functional on $\mathbb{F}_q[x]/f$. To be more specific, the spectra of gcd-graphs in both cases have an explicit description via Ramanujan sums. For the case of gcd-graphs over $\mathbb{Z}$, we refer readers to [6, Section 4]. For $\mathbb{F}_q[x]$, we will now recall the definition of the Ramanujan sums which describe the spectra of gcd-graphs.

Definition 2.1. (see [8, Section 6.2]) Let $f \in \mathbb{F}_q[x]$ be a monic polynomial. For each $g \in \mathbb{F}_q[x]$, the Ramanujan sum $c(g,f)$ is defined as follow

$$c(g,f) = c_{\psi}(g,f) = \sum_{a \in (\mathbb{F}_q[x]/f)^{\times}} \zeta_p^{\mathrm{Tr}(\psi(ga))}.$$

Here $\psi : \mathbb{F}_q[x]/f \to \mathbb{F}_q$ is a non-degenerate function on $\mathbb{F}_q[x]$ and $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map.

While the above definition looks rather complicated, we can show that $c(g,f)$ has a simple expression almost analogous to the case of Ramanujan sums over $\mathbb{Z}$. In particular, $c(g,f)$ does not depend on the choice of ψ as long as we make sure that it is non-degenerate. More precisely, by [8, Section 6.2] we have

$$c(g,f) = \mu(t)\frac{\varphi(f)}{\varphi(t)}, \quad \text{where} \quad t = \frac{f}{\gcd(f,g)}. \tag{2.1}$$

The following definition is inspired by the work [11] of Sander-Sander on gcd-graphs over $\mathbb{Z}$.

Definition 2.2. Let $\mathrm{Div}(f)$ be the set of proper monic divisors of f. Let $D \subset \mathrm{Div}(f)$. The spectral vector of the gcd-graph $G_f(D)$ is defined to be the following vector

$$\vec{\lambda}(f,D) = \left(\lambda_g(f,D)\right)_{g \in \mathbb{F}_q[x]/f}$$

where

$$\lambda_g(f,D) = \sum_{d \in D} c\left(g, \frac{f}{d}\right). \tag{2.2}$$

By the main result in [8, Section 6], the components of this spectral vector are precisely the eigenvalues of $G_f(D)$ counted with multiplicity. The following theorem is a direct analog of [11, Theorem 1.2]

Theorem 2.3. *Let D_1, D_2 be two proper subsets of $Div(f)$. Suppose that $G_f(D_1)$ and $G_f(D_2)$ have the same spectral vector. Then $D_1 = D_2$.*

To prove Theorem 2.3, we adapt the strategy employed in [12] which deals with gcd-graphs over $\mathbb{Z}$. More precisely, we will introduce and study a matrix similar to the one defined in [12]. Let $g, h \in \mathbb{F}_q[x]$ and $c(g,h)$ the Ramanujan sum. Let $C_f = (c(g,h))_{g,h \in \mathrm{Div}(f)}$. In [12, Theorem 1], the author solves the weak conjecture of Sander-Sander by showing that the determinant of C_n is not zero. In the case of $\mathbb{F}_q[x]$, we have a similar statement.

Proposition 2.4. *Let $|f|$ be the norm of f; i.e, $|f|$ is the order of the finite ring $\mathbb{F}_q[x]/f$. Then*

$$\det(C_f) = \pm |f|^{\frac{\tau(f)}{2}}.$$

Here $\tau(f)$ is the number of monic divisors of f. In particular $\det(C_f) \neq 0$.

Proof. We proceed by induction on the number of monic irreducible factors of f. If $\deg f = 0$ then $f = 1$ and $\det(C_1) = 1$. Now let $\deg f > 0$ and P a monic irreducible polynomial comprime to f. Let n be a positive integer. Let $f_1, f_2, \ldots, f_r$ be the monic divisors of f, here $r = \tau(f)$. We list the monic divisors of $P^n f$ as

$$f_1, f_2, \ldots, f_r, Pf_1, Pf_2, \ldots, Pf_r, \ldots, P^2 f_1, P^2 f_2, \ldots, P^2 f_r, \ldots, P^n f_1, P^n f_2, \ldots, P^n f_r.$$

Then $C_{P^n f}$ is an $(n+1) \times (n+1)$-block matrix

$$\begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0n} \\ C_{10} & C_{11} & \cdots & C_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n0} & C_{n1} & \cdots & C_{nn} \end{bmatrix},$$

where C_{ij} is a $r \times r$-matrix whose (k,l)-entry is

$$(C_{ij})_{kl} = c(P^i d_k, P^j d_l).$$

Clearly $C_{00} = C_f$. If $j \geq i+2$ then

$$c(P^i d_k, P^j d_l) = \frac{\varphi(P^j d_l)}{\varphi(\frac{P^j d_l}{\gcd(P^i d_k, P^j d_l)})} \mu(\frac{P^j d_l}{\gcd(P^i d_k, P^j d_l)}) = 0.$$

This is because $\frac{P^j d_l}{\gcd(P^i d_k, P^j d_l)}$ is divisible by P^2.

If $j = i+1$ then

$$\begin{aligned} c(P^i d_k, P^j d_l) &= \frac{\varphi(P^j d_l)}{\varphi(\frac{P^j d_l}{\gcd(P^i d_k, P^j d_l)})} \mu(\frac{P^j d_l}{\gcd(P^i d_k, P^j d_l)}) \\ &= \frac{\varphi(P^{i+1})}{\varphi(P)} \frac{\varphi(d_l)}{\varphi(\frac{d_l}{\gcd(d_k, d_l)})} \mu(P) \mu(\frac{d_l}{\gcd(d_k, d_l)}) \\ &= -|P|^i \frac{\varphi(d_l)}{\varphi(\frac{d_l}{\gcd(d_k, d_l)})} \mu(\frac{d_l}{\gcd(d_k, d_l)}) = -|P|^i c(d_k, d_l), \end{aligned}$$

and hence $C_{i,i+1} = -|P|^i C_f$.

If $j \leq i$ then

$$\begin{aligned} c(P^i d_k, P^j d_l) &= \frac{\varphi(P^j d_l)}{\varphi(\frac{P^j d_l}{\gcd(P^i d_k, P^j d_l)})}\mu(\frac{P^j d_l}{\gcd(P^i d_k, P^j d_l)}) \\ &= \frac{\varphi(P^j)\varphi(d_l)}{\varphi(\frac{d_l}{\gcd(d_k, d_l)})}\mu(\frac{d_l}{\gcd(d_k, d_l)}) \\ &= \varphi(P^j)c(d_k, d_l), \end{aligned}$$

and hence $C_{ij} = \varphi(P^j)C_f$. Thus

$$C_{P^n f} = \begin{bmatrix} C_f & -C_f & 0 & 0 & \cdots & 0 \\ C_f & \varphi(P)C_f & -|P|C_f & 0 & \cdots & 0 \\ C_f & \varphi(P)C_f & \varphi(P^2)C_f & -|P|^2C_f & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & & \\ C_f & \varphi(P)C_f & \varphi(P^2)C_f & \varphi(P^3)C_f & \cdots & -|P|^{n-1}C_f \\ C_f & \varphi(P)C_f & \varphi(P^2)C_f & \varphi(P^3)C_f & \cdots & \varphi(P^n)C_f \end{bmatrix}.$$

Subtracting the nth row from the $n+1$th row and noting that $\varphi(P^n) + |P|^{n-1} = |P^n|$, we get

$$\begin{aligned} \det(C_{P^n f}) &= \pm \det \begin{bmatrix} C_f & -C_f & 0 & 0 & \cdots & 0 \\ C_f & \varphi(P)C_f & -|P|C_f & 0 & \cdots & 0 \\ C_f & \varphi(P)C_f & \varphi(P^2)C_f & -|P|^2C_f & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & & \\ C_f & \varphi(P)C_f & \varphi(P^2)C_f & \varphi(P^3)C_f & \cdots & -|P|^{n-1}C_f \\ 0 & 0 & 0 & 0 & \cdots & |P|^nC_f \end{bmatrix} \\ &= \pm|P|^{rn}\det(C_f)\det \begin{bmatrix} C_f & -C_f & 0 & 0 & \cdots & 0 \\ C_f & \varphi(P)C_f & -|P|C_f & 0 & \cdots & 0 \\ C_f & \varphi(P)C_f & \varphi(P^2)C_f & -|P|^2C_f & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & & \\ C_f & \varphi(P)C_f & \varphi(P^2)C_f & \varphi(P^3)C_f & \cdots & \varphi(P^{n-1})C_f \end{bmatrix} \\ &= \cdots = \pm|P|^{rn}\det(C_f)|P|^{r(n-1)}\det(C_f)\cdots|P^r|\det(C_f) \\ &= \pm|P|^{\frac{n(n+1)}{2}\tau(f)}(\det C_f)^{n+1}. \end{aligned}$$

By the induction hypothesis

$$\det(C_{p^n f}) = \pm|P|^{\frac{n(n+1)}{2}\tau(f)}(|f|^{\frac{\tau(f)}{2}})^{n+1} = \pm(|P|^n|f|)^{\frac{(n+1)\tau(f)}{2}} = \pm|P^n f|^{\frac{\tau(P^n f)}{2}},$$

and we are done. □

We can now give a proof for Theorem 2.3.

Proof. Assume that $G_f(D_1)$ and $G_f(D_2)$ have the same spectral vector $\vec{\lambda}(f, D_1) = \vec{\lambda}(f, D_2)$. Then, the subvectors $(\lambda_g(f, D_1))_{g|f}$ and $(\lambda_g(f, D_2))_{g|f}$ indexed by divisors of f are also equal. For each $i \in \{1,2\}$, we define the following indicator vector of size $|\tau(f)| \times 1$

$$(v_i)_h = \begin{cases} 1 & \text{if } f/h \in D_i, \\ 0 & \text{if } f/h \notin D_i. \end{cases}$$

We then see that the vector $(\lambda_g(f, D_1))_{g|f}$ (respectively $(\lambda_g(f, D_2))_{g|f}$) are precisely $C_f v_1$ (respectively $C_f v_2$). Since C_f is invertible, we conclude that $v_1 = v_2$ and therefore, $D_1 = D_2$. □

3. GCD-GRAPHS ASSOCIATED WITH PRIME POWERS

In this section, we study the gcd-graphs $G_f(D)$ in the case f is a prime power; i.e, $f = P^k$ where $P \in \mathbb{F}_q[x]$ is an irreducible polynomial and $k \geq 1$. In this case, a subset D of $\mathrm{Div}(P^k)$ can be written uniquely in the following form $D = \{P^{k_1}, P^{k_2}, \ldots, P^{k_s}\}$ where

$$0 \leq k_1 < k_2 < \ldots < k_s < k.$$

We will fix this notation throughout this section.

3.1. Graph theoretic properties of $G_{P^k}(D)$.

We first discuss some fundamental graph-theoretic properties of $G_{P^k}(D)$. Recall that for each $f \in \mathbb{F}_q[x]$, $|f| = q^{\deg(f)}$ is the order of the finite ring $\mathbb{F}_q[x]/f$.

Proposition 3.1. *$G_{P^k}(D)$ has exactly $|P|^{k_1}$ connected components and each component is isomorphic to $G_{P^{k-k_1}}(D')$ where*

$$D' = \{1, P^{k_2-k_1}, \ldots, P^{k_s-k_1}\}.$$

In particular, $G_{P^k}(D)$ is connected if and only if $k_1 = 0$.

Proof. By definition, the cosets $\{g + P^{k_1}(\mathbb{F}_q[x]/P^k)\}$ where g runs over $\mathbb{F}_q[x]/P^{k_1}$ are mutually unconnected. Furthermore, by [8, Lemma 5.4], each of these cosets is isomorphic to $G_{P^{k-k_1}}(D')$ which is connected by [8, Corollary 3.4]. □

Proposition 3.2. *$G_{P^k}(D)$ is a bipartite graph if and only the following conditions hold*

(1) $\mathbb{F}_q = \mathbb{F}_2$.
(2) $\deg(P) = 1$; *namely either $P = x$ or $P = x + 1$.*
(3) $D = \{1\}$.

Proof. Suppose that $G_{P^k}(D)$ is a bipartite graph. By [8, Corollary 4.2], we must have $\mathbb{F}_q = \mathbb{F}_2$ and $\gcd(P^k, x(x+1)) \neq 1$. Since P is irreducible, we conclude that either $P = x$ or $P = x + 1$. By [8, Theorem 4.3], we also know that for each $1 \leq i \leq s$, $\gcd(P^k, x(x+1)) \nmid P^{k_i}$. This happens only if $k_i = 0$; or equivalently $D = \{1\}$. Conversely, if all of the above conditions are satisfied, then by [8, Theorem 4.3] $G_{P^k}(D)$ is bipartite. We can in fact show a concrete bipartite partition of $G_{P^k}(D)$ as follows

$$V(G_{P^k}(D)) = A_0 \bigsqcup A_1,$$

where

$$A_0 = \{h \in \mathbb{F}_q[x]/P^k \text{ such that } P \mid h\},$$

and

$$A_1 = \{h \in \mathbb{F}_q[x]/P^k \text{ such that } P \nmid h\}.$$

□

We now discuss the decomposition of $G_{P^k}(D)$ into the wreath product (also known as the lexicographic product) of simpler graphs. First, we recall the definition of the wreath product and homogeneous sets.

Definition 3.3. Let Γ, Δ be two graphs. The wreath product of Γ and Δ is the graph $\Gamma * \Delta$ equipped with the following data

(1) The vertex set of $\Gamma * \Delta$ is the Cartesian product $V(\Gamma) \times V(\Delta)$,
(2) (x, y) and (x', y') are adjacent in $\Gamma * \Delta$ if either $(x, x') \in E(\Gamma)$ or $x = x'$ and $(y, y') \in E(\Delta)$.

Definition 3.4. Let G be a graph. A homogeneous set in G is a set X of vertices of G such that every vertex in $V(G) \setminus X$ is adjacent to either all or none of the vertices in X. A homogenous set X is said to be non-trivial if $2 \leq X < |V(G)|$.

As shown in [4, Section 3], the existence of a homogenous set in a Cayley graph is almost equivalent to the existence of a decomposition of G into a wreath product. In [8, Section 5], we describe the necessary and sufficient conditions for the existence of homogeneous sets in the gcd-graphs $G_f(D)$ over $\mathbb{F}_q[x]$. When $f = P^k$, the situation is relatively simple. In fact, by [8, Theorem 5.5], we have the following.

Theorem 3.5. *Let $I = \langle P^{k-1} \rangle$ be the ideal of $\mathbb{F}_q[x]/P^k$ generated by P^{k-1}. Then I is an homogeneous set in $G_{P^k}(D)$. Furthermore*

$$G_{P^k}(D) \cong \begin{cases} G_{P^{k-1}}(D_1) * K_{|P|} & \text{if } P^{k-1} \in D, \\ G_{P^{k-1}}(D_1) * E_{|P|} & \text{if } P^{k-1} \notin D. \end{cases}$$

Here $D_1 = \{P^{k_1}, \ldots, P^{k_{s-1}}\}$, $*$ is the wreath product, K_m (respectively E_m) the complete graph (respectively the co-complete graph) on m nodes.

We discuss various consequences of Theorem 3.5. First, it is known that the wreath product $G_1 * G_2$ is perfect if and only if both G_1, G_2 are perfect (see [7]). Since complete and co-complete graphs are perfect, by mathematical induction, we conclude from Theorem 3.5 that.

Corollary 3.6. *$G_{P^k}(D)$ is a perfect graph.*

Next, we discuss the clique number of $G_{P^k}(D)$. Because the clique number $\omega(G)$ of a graph G behaves well with respect to wreath products: $\omega(G_1 * G_2) = \omega(G_1)\omega(G_2)$, we have the following conclusion.

Corollary 3.7. *The clique and chromatic numbers of $G_{P^k}(D)$ are $|P|^{|D|} = |P|^s$.*

Proof. By Theorem 3.5, we know that

$$G_{P^k}(D) \cong \begin{cases} G_{P^{k-1}}(D_1) * K_{|P|} & \text{if } P^{k-1} \in D, \\ G_{P^{k-1}}(D_1) * E_{|P|} & \text{if } P^{k-1} \notin D. \end{cases}$$

Here $D_1 = \{P^{k_1}, \ldots, P^{k_{s-1}}\}$. Since the clique number behaves well with respect to the wreath product, we have

$$\omega(G_{P^k}(D)) = \begin{cases} |P|\omega(G_{P^{k-1}}(D_1)) & \text{if } P^{k-1} \in D, \\ \omega(G_{P^{k-1}}(D_1)) & \text{if } P^{k-1} \notin D. \end{cases}$$

By induction, we conclude that $\omega(G_{P^k}(D)) = |P|^s$. Since $G_{P^k}(D)$ is perfect, its chromatic number is also equal to $|P|^s$. □

Finally, the wreath product also behaves well with respect to taking complement: $(G_1 * G_2)^c = (G_1)^c * (G_2)^c$. Therefore, by an identical argument, we also have.

Corollary 3.8. *The indepedent number of $G_{P^k}(D)$ is $|P|^{k-|D|} = |P|^{k-s}$.*

3.2. **Spectral properties of $G_{P^k}(D)$.** We now focus on the spectral properties of $G_{P^k}(D)$. While several of the results in this section are similar to those in [10, 11], many are new, thanks to insights inspired by the experimental data that we generate for this project. We first have the following observation about Ramanujan sums.

Lemma 3.9. *Suppose that $P \in \mathbb{F}_q[x]$ be a monic irreducible polynomial. Let m, k be two non-negative integers. Then*

$$c(P^m, P^k) = \begin{cases} \varphi(P^k) & \text{if } m \geq k, \\ -|P|^m & \text{if } k - m = 1, \\ 0 & \text{if } k - m \geq 2. \end{cases}$$

Proof. This follows directly from Eq. (2.1). □

Let us introduce the following notation which is inspired by a counterpart over $\mathbb{Z}$ (see [11, Theorem 1.1]).

Definition 3.10. We define the following function

$$\chi(P^k, D, t) = \begin{cases} 1 & \text{if } k_i = k - t - 1 \text{ for some } 1 \leq i \leq s, \\ 0 & \text{else.} \end{cases}$$

With this notation, we can now calculate the spectrum of $G_{P^k}(D)$ explicitly. The following statement is a direct analog of [11, Theorem 1.1] in the function fields setting.

Theorem 3.11. *Let $g \in \mathbb{F}_q[x]/P^k$ and $P^t = \gcd(P^k, g)$. Let $\lambda_g(P^k, D)$ be the eigenvector of $G_{P^k}(D)$ as described in Eq. (2.2). Then we have the following.*

$$\begin{aligned} \lambda_g(P^k, D) = \lambda_{P^t}(P^k, D) &= -\chi(P^k, D, t)|P|^t + \sum_{k_i \geq k-t} \varphi(P^{k-k_i}) \\ &= -\chi(P^k, D, t)|P|^t + (|P| - 1) \sum_{k_i \geq k-t} |P|^{k-k_i-1}. \end{aligned}$$

Proof. By Eq. (2.2), we know that $c(g, f) = c(\gcd(f, g), f)$ for all f and g. In particular, we have $\lambda_g(P^k, D) = \lambda_{P^t}(P^k, D)$. We then have

$$\lambda_{P^t}(P^k, D) = \sum_{i=1}^{s} c(P^t, P^{k-k_i}).$$

By Lemma 3.9 and the definition of $\chi(P^k, D, t)$ we conclude that

$$\lambda_{P^t}(P^k, D) = -\chi(P^k, D, t)|P|^t + \sum_{k_i \geq k-t} \varphi(P^{k-k_i}) = -\chi(P^k, D, t)|P|^t + (|P|-1) \sum_{k_i \geq k-t} |P|^{k-k_i-1}.$$

□

We now use Theorem 3.11 to describe various arithmetic and algebraic properties of the spectrum of $G_{P^k}(D)$.

3.2.1. ***Congruence of eigenvalues of*** $G_{P^k}(D)$. In this section, by utilizing Theorem 3.11, we investigate some congruence properties of eigenvalues of $G_{P^k}(D)$. We then give an explicit criterion for a number m to be an eigenvalue of $G_{P^k}(D)$ for some choice of D. In particular, we show that every integer m can be realized as an eigenvector of $G_{P^k}(D)$ for appropriate choices of P and D. We start this section with an observation which is suggested by our numerical data.

Proposition 3.12. *Let $\lambda \in \mathbb{Z}$ be an eigenvalue of $G_{P^k}(D)$. Then*

$$\lambda \equiv \begin{cases} 0 \pmod{|P|-1} & \text{if } \lambda \geq 0 \\ -1 \pmod{|P|-1} & \text{if } \lambda < 0. \end{cases}$$

Proof. Suppose that $\lambda = \lambda_g(P^k, D)$ for some D where we keep the same notation as in Theorem 3.11. We have

$$\lambda_g(P^k, D) = \lambda_{P^t}(P^k, D) = -\chi(P^k, D, t)|P|^t + \sum_{k_i \geq k-t} \varphi(P^{k-k_i}),$$

We have

$$0 \leq \sum_{k > k_i \geq k-t} \varphi(P^{k-k_i}) \leq \sum_{i=1}^{t} \varphi(P^i) = (|P|-1)\frac{|P|^t - 1}{|P|-1} = |P|^t - 1.$$

Additionally, since $\chi(P^k, D, t) \in \{0, 1\}$, we conclude that $\lambda \geq 0$ if and only $\chi(P^k, D, t) = 0$. Therefore

$$\lambda = \begin{cases} \sum_{k_i \geq k-t} \varphi(P^{k-k_i}) = (|P|-1) \sum_{k_i \geq k-t} |P|^{k-k_i-1} & \text{if } \lambda \geq 0. \\ -|P|^t + \sum_{k_i \geq k-t} \varphi(P^{k-k_i}) = -|P|^t + (|P|-1) \sum_{k_i \geq k-t} |P|^{k-k_i-1} & \text{if } \lambda < 0. \end{cases} \tag{3.1}$$

We then conclude that

$$\lambda \equiv \begin{cases} 0 \pmod{|P|-1} & \text{if } \lambda \geq 0 \\ -1 \pmod{|P|-1} & \text{if } \lambda < 0. \end{cases}$$

□

We have a direct corollary of Proposition 3.12.

Corollary 3.13. *Let λ be an integer satisfying the congruence condition*

$$\lambda \equiv \begin{cases} 0 \pmod{|P|-1} & \text{if } \lambda \geq 0 \\ -1 \pmod{|P|-1} & \text{if } \lambda < 0. \end{cases}$$

Let λ_0 be a non-negative integer defined by the following rule

$$\lambda_0 = \begin{cases} \frac{\lambda}{|P|-1} \pmod{|P|-1} & \text{if } \lambda \geq 0. \\ \frac{|\lambda+1|}{|P|-1} \pmod{|P|-1} & \text{if } \lambda < 0. \end{cases}.$$

Then m is an eigenvalue of $G_{P^k}(D)$ for some k and D if and only each digit in the base $|P|$ representation of m_0 is either 0 or 1.

Proof. Let us keep the same notation as in the proof of Theorem 3.11. Then, we have

$$\lambda = \begin{cases} (|P|-1)\sum_{k_i \geq k-t} |P|^{k-k_i-1} & \text{if } \lambda \geq 0. \\ -|P|^t + (|P|-1)\sum_{k_i \geq k-t} |P|^{k-k_i-1} & \text{if } \lambda < 0. \end{cases}.$$

Consequently, we have

$$\lambda_0 = \begin{cases} \sum_{k_i \geq k-t} |P|^{k-k_i-1} & \text{if } \lambda \geq 0. \\ \sum_{i=0}^{t-1} |P|^i - \sum_{k_i \geq k-t} |P|^{k-k_i-1} & \text{if } \lambda < 0. \end{cases}.$$

We can see that the statement follows directly from this formula. □

If $|P| = 2$ then we can see that the congruence condition mentioned in Proposition 3.12 is trivial. As a result, we have the following.

Proposition 3.14. *For each integer m, there exists an integer k and a subset $D \subset \mathrm{Div}(P^k)$ with $P = x \in \mathbb{F}_2[x]$ such that m is an eigenvalue of $G_{P^k}(D)$.*

3.2.2. ***Some special eigenvalues of*** $G_{P^k}(D)$. We now discuss the existence of small eigenvalues. Specifically, for each $\lambda \in \{-1, 0, 1\}$ we find the necessary and sufficient conditions for λ to be an eigenvalue of $G_{P^k}(D)$. Quite surprisingly, we will see that 0 and -1 cannot simultaneously an eigenvalue of $G_{P^k}(D)$ (see Corollary 3.17). We start our discussion with the eigenvalue 0.

Proposition 3.15. *The following statements are equivalent*

(1) $P^{k-1} \notin D$.
(2) 0 is an eigenvalue of $G_{P^k}(D)$.

Consequently, there are exactly 2^{k-1} graphs in the family $G_{P^k}(D)$ that has 0 as an eigenvalue.

Proof. Let us first show that (1) implies (2). Let $\lambda = \lambda_{P^0}(P^k, D)$. Since $P^{k-1} \notin D$ we know that $\chi(P^k, D, 0) = 0$. Consequently, by Theorem 3.11, we conclude that $\lambda = 0$.

Conversely, let us assume that $\lambda = \lambda_{P^t}(P^k, D) = 0$ is an eigenvalue of $G_{P^k}(D)$. Suppose to the contrary that $P^{k-1} \in D$. By the proof of Proposition 3.12 we know that $\chi(P^k, D, t) = 0$ and

$$\sum_{k_i \geq k-t} |P|^{k-k_i-1} = 0.$$

Since $P^{k-1} \in D$, this implies that $t = 0$. However, this would implie that $\chi(P^k, D, 0) = 1$ which is a contradiction. We conclude that $P^{k-1} \in D$. □

For the eigenvalue -1, we have the following proposition.

Proposition 3.16. *The following statements are equivalent*

(1) $P^{k-1} \in D$.
(2) -1 *is an eigenvalue of* $G_{P^k}(D)$.

Consequently, there are exactly 2^{k-1} *graphs in the family* $G_{P^k}(D)$ *that has* -1 *as an eigenvalue.*

Proof. Let us first show that (1) implies (2). Since $P^{k-1} \in D$, we conclude that $\chi(P^k, D, 0) = 1$. By Theorem 3.11, we conclude that $\lambda_{P^0}(P^k, D) = -1$.

Conversely, suppose that $\lambda = \lambda_{P^t}(P^k, D) = -1$ is an eigenvalue of $G_{P^k}(D)$. Then $\chi(P^k, D, t) = 1$ and $-1 = \lambda = -|P|^t + (|P| - 1) \sum_{k_i \geq k-t} |P|^{k-k_i-1}$. This implies that

$$\sum_{i=0}^{t-1} |P|^i = \sum_{k_i \geq k-t} |P|^{k-k_i-1}.$$

We conclude that $P^{k-1} \in D$. □

By combining Proposition 3.16 and Proposition 3.15 we have the following rather surprising corollary.

Corollary 3.17. *For each D, exactly one of the values* $\lambda = 0$ *or* $\lambda = -1$ *is an eigenvalue of* $G_{P^k}(D)$.

Our numerical data suggests that 1 is an eigenvalue of $G_{P^k}(D)$ under some very special conditions. We remark that by Proposition 3.1, we only need to consider the case $G_{P^k}(D)$ is connected; i.e, $1 \in D$.

Proposition 3.18. *Suppose that* $1 \in D$. *Then* 1 *is an eigenvalue of* $G_{P^k}(D)$ *if and only if the following conditions hold*

(1) $\mathbb{F}_q = \mathbb{F}_2$.
(2) $\deg(P) = 1$.
(3) $P^{k-1} \in D$.
(4) $P^{k-2} \notin D$.

Proof. First, let us assume that $\lambda = 1$ is an eigenvalue of $G_{P^k}(D)$. Then the congruence relation Proposition 3.12 implies that $|P| - 1$ is a divisor of λ. This happens only if $2 = |P| = q^{\deg(P)}$. We conclude that $q = 2$ and $\deg(P) = 1$. Furthemore, by Eq. (3.1), we can find t such that $\chi(P^k, D, t) = 0$ and

$$1 = \sum_{k_i \geq k-t} |P|^{k-k_i-1}.$$

This happens only when $t = 1$ and $P^{k-1} \in D$. Since $\chi(P^k, D, 1) = 0$, we conclude that $P^{k-2} \notin D$. For the other direction, we can see that if all of the above conditions are satisfied then $\lambda_{P^1}(P^k, D) = 1$. □

3.2.3. ***Some estimations on the eigenvalues of*** $G_{P^k}(D)$. In this section, we discuss some rather simple estimations on the eigenvalues of $G_{P^k}(D)$. This section provides a direct analog of [11, Section 3]. Keeping the notation as in Theorem 3.11, we have the following (compare with [11, Corollary 2.1]).

Proposition 3.19. *Let* $0 \leq u \leq v < k$. *Then*

$$|\lambda_{P^u}(P^k, D)| \leq |\lambda_{P^v}(P^k, D)|.$$

Furthermore, $\lambda_{P^k}(P^k, D)$ *is the degree of* $G_{P^k}(D)$ *which is also its largest eigenvalues.*

Proof. This statement follows directly from Eq. (3.1) and the fact that for each $t \geq 1$

$$|P|^t > (|P|-1)\sum_{i=0}^{t-1} |P|^i.$$

□

Remark 3.20. The proof for [11, Corollary 2.1] is somewhat more complicated because the authors allow loops in $G_{P^k}(D)$; i.e, they consider the possibility that $P^k \in D$.

We now discuss some corollaries of Proposition 3.19. The first corollary concerns the largest eigenvalues in $G_f(D)$.

Corollary 3.21. *Let* D_1, D_2 *be two subsets of* $\mathrm{Div}(P^k)$ *such that* $G_{P^k}(D_1)$ *and* $G_{P^k}(D_2)$ *has the same largest eigenvalue. Then* $D_1 = D_2$.

Proof. Let $D_1 = \{k_1, k_2, \ldots, k_s\}$ and $D_2 = \{h_1, h_2, \ldots, h_t\}$. Suppose that $G_{P^k}(D_1)$ and $G_{P^k}(D_2)$ have the same largest eigenvalue. Then $\sum_{i=1}^{s} |P|^{k-k_i} = \sum_{i=1}^{t} |P|^{k-h_i}$. Let u be the last index such that $k_u \neq h_u$. Then we

$$\sum_{i=u}^{s} |P|^{k-k_i} = \sum_{i=u}^{t} |P|^{k-h_i}.$$

If $u = s$ or $u = t$, then we must have $s = t$ and $D_1 = D_2$. Otherwise, suppose that $u < \min(s, t)$. Without loss of generality, we can also assume that $k - k_u > k - h_u$. We have then

$$\sum_{i=u}^{s} |P|^{k-k_i} \geq |P|^{k-k_u} \geq |P|^{k-h_u+1} > \sum_{i=0}^{k-k_u} |P|^i \geq \sum_{i=u}^{t} |P|^{k-h_i}.$$

We conclude that $D_1 = D_2$. □

Remark 3.22. Similar to the case of gcd-graphs over $\mathbb{Z}$ (see [11, Section1] for an example in this case), Corollary 3.21 is false if f is not a prime power. For example, let $f = x^2(x+1) \in \mathbb{F}_3[x]$, $D_1 = \{1\}$ and $D_2 = \{x, x+1, x(x+1)\}$. Then the characteristic polynomials of $G_f(D_1)$ and $G_f(D_2)$ are respectively

$$(x-12)(x-3)^4(x+6)^4 x^{18},$$

and

$$(x-12)(x-6)^2(x-3)^2(x+3)^{10} x^{12}.$$

We see that $G_f(D_1)$ and $G_f(D_2)$ have the same largest eigenvalue but they are not isomorphic.

We discuss another corollary of Proposition 3.19 which we find by investigating our numerical data.

Corollary 3.23. *Suppose that $D \neq \emptyset$. Then $G_{P^k}(D)$ has at least two distinct eigenvalues. Furthermore, $G_{P^k}(D)$ has exactly two eigenvalues if and only if*

$$(k_1, k_2, \ldots, k_s) = (k_1, k_1 + 1, \ldots, k - 1).$$

In this case, $G_f(D)$ is the disjont union of $|P|^{k_1}$ copies of K_m where $m = |P|^{k-k_1}$.

Proof. We know that the largest eigenvalues of $G_{P^k}(D)$ is its degree which is

$$\lambda_{P^k}(P^k, D) = (|P| - 1) \sum_i |P|^{k-k_i-1} > 0.$$

Furthermore, since the sum of all eigenvalues of $G_{P^k}(D)$ is 0, there must be an eigenvalue that is negative. Therefore, $G_{P^k}(D)$ must have at least two distinct eigenvalues. Note that, this part holds true for all undirected simple graphs.

Suppose that $G_{P^k}(D)$ has exactly two eigenvalues. We need to show that $k_{i+1} - k_i = 1$ for $1 \leq i \leq s - 1$. Suppose that is not the case. Then, we can find $1 \leq i \leq s - 1$ such that $k_{i+1} - k_i > 1$. We then see that $\chi(P^k, D, k - k_{i+1}) = 0$ since there is no j such that $k_j = k - (k - k_{i+1}) - 1 = k_{i+1} - 1$. We then see that

$$\lambda_{P^{k-k_{i+1}}}(P^k, D) = (|P| - 1) \sum_{j \geq i+1} |P|^{k-k_j-1}.$$

Since $\lambda_{P^{k-k_{i+1}}}(P^k, D) > 0$ and $G_{P^k}(D)$ has exactly two eigenvalues $\lambda_{P^{k-k_{i+1}}}(P^k, D) = \lambda_{P^k}(P^k, D)$. Therefore $\sum_i |P|^{k-k_i-1} = \sum_{j \geq i+1} |P|^{k-k_j-1}$, which is impossible. We conclude that $k_{i+1} - k_i = 1$ for all $1 \leq i \leq s - 1$.

□

We now conclude this section with a main theorem which says that D is determined by the spectrum of $G_{P^k}(D)$.

Theorem 3.24. *Let D_1, D_2 be two subsets of $\operatorname{Div}(P^k)$. Then the following statements are equivalent*

(1) $D_1 = D_2$.
(2) $G_{P^k}(D_1)$ and $G_{P^k}(D_2)$ are isomorphic.
(3) $G_{P^k}(D_1)$ and $G_{P^k}(D_2)$ are isospectral.

Proof. By definition (1) $\implies$ (2) $\implies$ (3). Let us show that (3) $\implies$ (1). In fact, suppose that $G_{P^k}(D_1)$ and $G_{P^k}(D_2)$ are isospectral. Then, in particular, they share the same largest eigenvalue. By Corollary 3.21, we must have $D_1 = D_2$. □

4. Isomorphic gcd-graphs

4.1. **Isomorphic unitary Cayley graphs in the family $\mathbb{F}_q[x]/f$.** Let R be a finite commutative ring. The unitary Cayley G_R graph on R is defined as the graph whose vertex set is R and two vertices a, b are adjacent if $a - b \in R^\times$. Let $\operatorname{Rad}(R)$ be the Jacobson radical of R and $R^{ss} = R/\operatorname{Rad}(R)$ be the semi-simplification of R. In [4, Section 4], it is shown that $\operatorname{Rad}(R)$ is a homogeneous set in R. Furthermore, G_R is isomorphic to the wreath product $G_{R^{ss}} * E_m$

where $m = |\mathrm{Rad}(R)|$. In [5, Theorem 5.3], Kiani and Aghaei show for two commutative rings R_1, R_2, if $G_{R_1} \cong G_{R_2}$ then $R_1^{\mathrm{ss}} \cong R_2^{\mathrm{ss}}$. Combining these two facts, we have the following.

Proposition 4.1. *Suppose that R_1, R_2 are two commutative rings. Then the following conditions are equivalent*

(1) $G_{R_1} \cong G_{R_2}$
(2) $|R_1| = |R_2|$ *and* $R_1^{ss} \cong R_2^{ss}$.

We will use Proposition 4.1 to classify isomorphism classes of unitary graphs in the family $\mathbb{F}_q[x]/f$ where f is a monic polynomial of fixed degree n. Let $\mathrm{rad}(f)$ be the radical of f; i.e. the product of all distinct prime factors of f. Then the Jacobson radical of $\mathbb{F}_q[x]/f$ is precisely the ideal generated by $\mathrm{rad}(f)$. Consequently

$$(\mathbb{F}_q[x]/f)^{\mathrm{ss}} \cong \mathbb{F}_q[x]/\mathrm{rad}(f).$$

By Proposition 4.1, the isomorphism class of $G_f(\{1\})$ depends on $\mathrm{rad}(f)$ only. We remark, however, that two polynomials with different radicals can still give rise to isomorphic unitary graphs. For example, take the following examples $f_1 = x^2, f_2 = (x+1)^2$. Then $\mathrm{rad}(f_1) = x, \mathrm{rad}(f_2) = x+1$ but

$$\mathbb{F}_q[x]/\mathrm{rad}(f_1) \cong \mathbb{F}_q[x]/\mathrm{rad}(f_2) \cong \mathbb{F}_q.$$

Consequently $G_{f_1}(\{1\}) \cong G_{f_2}(\{1\})$ even though f_1, f_2 have different radicals. In general, Galois theory for $\mathbb{F}_q$ implies that if g_1 and g_2 are two irreducible polynomials of the same degree, then $\mathbb{F}_q[x]/g_1 \cong \mathbb{F}_q[x]/g_2$. Motivated by this observation, we introduce the following definition.

Definition 4.2. Let $f \in \mathbb{F}_q[x]$ be a monic polynomial of degree n. We define the factorization type of f as the n-tuple $(a_1, a_2, \ldots, a_n)$ where a_i is the number of distinct irreducible factors of f of degree i.

Example 4.3. Let $f = x^2(x+1)(x^2+1) \in \mathbb{F}_3[x]$. Then, the degree of f is 5 and the factorization type of f is $(2, 1, 0, 0, 0)$.

By Proposition 4.1 and the previous discussion, we have the following.

Proposition 4.4. *Let $f_1, f_2 \in \mathbb{F}_q[x]$ be two monic polynomial. The unitary graphs $G_{f_1}(\{1\}$ and $G_{f_2}(\{1\})$ are isomorphic if and only if $\deg(f_1) = \deg(f_2)$ and f_1, f_2 have the same facotorizatin type.*

We will now use Proposition 4.4 to study the number of non-isomorphic classes of unitary graphs of the form $G_f(\{1\})$ where $f \in \mathbb{F}_q[x]$ and $\deg(f) = n$. Table 1 shows this number for various values of n and q. As we can observe from this dataset, once we fix n, the number of isomorphism classes seems to stabilize when q gets bigger. In fact, we have the following proposition.

Proposition 4.5. *Let n be a fixed number. Then, there exist two constants q_n, C_n depending only on n such that if $q \geq q_n$, then the number of non-isomorphic classes of unitary graphs of the form $G_f(\{1\})$ where $f \in \mathbb{F}_q[x]$ and $\deg(f) = n$ is exactly C_n.*

n \ q	2	3	4	5	7	8	9	11
1	1	1	1	1	1	1	1	1
2	3	3	3	3	3	3	3	3
3	4	5	5	5	5	5	5	5
4	7	9	10	10	10	10	10	10
5	9	12	13	14	14	14	14	14
6	15	22	24	25	26	26	26	26

TABLE 1. The number of isomorphism classes of unitary graphs in the family $\mathbb{F}_q[x]/f$ where $\deg(f) = n$ for various values of n and q

In order to provide a proof for Proposition 4.5, we recall that for fixed m, q, the number $S(m, q)$ of irreducible polynomials of degree m over $\mathbb{F}_q[x]$ is given by the following Gauss's formula (see [3] for a proof of this formula using the inclusion-exclusion principle)

$$\frac{1}{m}\sum_{d|m}\mu(m/d)q^d. \tag{4.1}$$

By Eq. (4.1), we see that for fixed n, $S(m, q)$ is asymptotically equivalent to q^m for all $m \leq n$. In particular, $S(q, m) \geq n$ for all q sufficiently large. We can now give a proof for Proposition 4.5

Proof. Let $f \in \mathbb{F}_q[x]$ be a monic polynomial of degree n and $(a_1, a_2, \ldots, a_n)$ be a factorization type. Then we have

$$\sum_{m=1}^{n} ma_m \leq n.$$

In particular, this shows that $a_m \leq n$ for all $1 \leq m \leq n$. We can find a number q_n such that for $q \geq q_n$ we have $n \leq S(m, q)$. Consequently, $a_m \leq S(m, q)$ as well. As long as this condition is satisfied, the number of ways to "pick" a_m distinct polynomials from the set of all irreducible polynomials over $\mathbb{F}_q[x]$ of degree m is independent of q. □

Remark 4.6. Table 1 suggests that we can in fact take $q_n = n$. Furthermore, if we fix n, then the number of isomorphism classes appears to increase when q increases. While both of these observations turn out to be correct, the proof is lengthy and somewhat tangential to the main focus of this paper. We plan to address these properties in a forthcoming work in connection with necklace polynomials [14].

4.2. **Isomorphic gcd-graphs.** In the previous section, we discuss the necessary and sufficient conditions for two unitary Cayley graphs to be isomorphic. In this section, we extend this further to the case of gcd-graphs. The case f is a prime power is studied exclusively in Section 3 so we will focus on the case f has at least two distinct factors in this section. Since many of our insights arise from analyzing experimental data, we will begin this section with a concrete numerical example.

Example 4.7. Let $f = x(x+1) \in \mathbb{F}_3[x]$. Table 2 describe the isomorphism classes in the family of $G_f(D)$ where D runs over the collection of subsets of $\text{Div}(f)$. We remark that we group these graphs by their characteristic polynomials; i.e. by isospectral classes, but we have checked that elements in the same class are isomorphic as well. This is consistent with a conjecture of So for gcd-graphs over $\mathbb{Z}$ saying that two gcd graphs $G_f(D_1)$ and $G_f(D_2)$ are isomorphic if and only if they are isospectral (see [13]).

D	**Characteristic polynomial of** $G_f(D)$
[]	x^9
$[1], [x, x+1]$	$(x-4)(x-1)^4(x+2)^4$
$[x], [x+1]$	$(x-2)^3(x+1)^6$
$[1, x], [1, x+1]$	$(x-6)(x+3)^2x^6$
$[1, x, x+1]$	$(x-8)(x+1)^8$

TABLE 2. Isomorphism classes in the family $G_f(D)$

Looking closely at the data, we have the following observation. While the isomorphism between $G_f(x)$ and $G_f(x+1)$ as well as the isomorphism between $G_f(1,x)$ and $G_f(1,x+1)$ are evident, the isomorphism between $G_f(1)$ and $G_f(x,x+1)$ is less so. In fact, this isomorphism appears like a coincidence. In fact, let us consider $f = x(x+1) \in \mathbb{F}_q[x]$. Then the degree $G_f(1)$ is $(q-1)^2$ and the degree of $G_f(x,x+1) = 2(q-1)$. These two degrees are equal only in the case $q = 3$.

We discuss a more complicated example.

Example 4.8. Let $f = x^2(x+1) \in \mathbb{F}_3[x]$. As expected, there are more isomorphism classes in this family. We remark, however, that similar to the previous example some isomorphisms depend on the fact that $q = 3$. More precisely, if we let $f = x^2(x+1) \in \mathbb{F}_5[x]$, then we no longer have an isomorphism between $G_f([1,x,x+1])$ and $G_f([1,x+1,x^2,x(x+1)])$. On the other hand, some isomorphisms persist when we change q. For example, using the Python library Networkx we find the following two phenomena.

(1) For every $q \leq 5$ and $f = x^2(x+1) \in \mathbb{F}_q[x]$, $G_f([1,x,x^2])$ and $G_f([1,x+1])$ are isomorphic.
(2) For every $q \leq 5$ and $f = x^2(x+1) \in \mathbb{F}_q[x]$, $G_f([1,x+1,x^2])$ and $G_f([1,x+1,x(x+1)])$ are also isomorphic.

We try other various examples and find the following statement which explains the first phenomena described in Example 4.8.

Proposition 4.9. *Let $f = f_1f_2$ such that $\gcd(f_1,f_2) = 1$. Assume further $\operatorname{rad}(f_1)$ and $\operatorname{rad}(f_2)$ have the same factorization type. Let*

$$D_1 = \operatorname{Div}(f_1) \cup \{f_1\},$$

and

$$D_2 = \operatorname{Div}(f_2) \cup \{f_2\}.$$

Then $G_f(D_1)$ and $G_f(D_2)$ are isomorphic. In fact, they are both isomorphic to the wreath product $G_{\mathbb{F}_q[x]/\operatorname{rad}(f_1)}(\{1\}) * E_m$ where $m = \frac{|f|}{|\operatorname{rad}(f_1)|}$.

D	Characteristic polynomial of $G_f(D)$
$[]$	x^{27}
$[1]$, $[x, x+1, x^2]$	$(x-12)(x-3)^4(x+6)^4x^{18}$
$[x]$, $[x^2, x(x+1)]$	$(x-4)^3(x-1)^{12}(x+2)^{12}$
$[1, x]$	$(x-16)(x+8)^2(x+2)^8(x-1)^{16}$
$[x+1]$, $[x, x^2]$, $[x, x(x+1)]$	$(x-6)^3(x+3)^6x^{18}$
$[1, x+1]$, $[1, x, x^2]$	$(x-18)(x+9)^2x^{24}$
$[x, x+1]$	$(x-10)(x-4)^2(x+5)^4(x+2)^6(x-1)^{14}$
$[1, x, x+1]$, $[1, x+1, x^2, x(x+1)]$	$(x-22)(x+5)^2(x-1)^{12}(x+2)^{12}$
$[x^2]$, $[x(x+1)]$	$(x-2)^9(x+1)^{18}$
$[1, x^2]$	$(x-14)(x+4)^2(x+7)^2(x-2)^{10}(x+1)^{12}$
$[x+1, x^2]$	$(x-8)(x-5)^2(x+4)^4(x-2)^6(x+1)^{14}$
$[1, x+1, x^2]$, $[1, x+1, x(x+1)]$, $[1, x, x^2, x(x+1)]$	$(x-20)(x+7)^2(x-2)^6(x+1)^{18}$
$[1, x, x+1, x^2]$, $[1, x, x+1, x(x+1)]$	$(x-24)(x+3)^8x^{18}$
$[1, x(x+1)]$, $[x, x+1, x^2, x(x+1)]$	$(x-14)(x-5)^4(x+4)^4(x+1)^{18}$
$[1, x, x(x+1)]$	$(x-18)(x+6)^2(x-3)^4(x+3)^6x^{14}$
$[x+1, x(x+1)]$, $[x, x^2, x(x+1)]$	$(x-8)^3(x+1)^{24}$
$[x, x+1, x(x+1)]$	$(x-12)(x-6)^2(x-3)^2(x+3)^{10}x^{12}$
$[1, x^2, x(x+1)]$	$(x-16)(x+5)^2(x-4)^4(x-1)^6(x+2)^{14}$
$[x+1, x^2, x(x+1)]$	$(x-10)(x-7)^2(x-1)^8(x+2)^{16}$
$[1, x, x+1, x^2, x(x+1)]$	$(x-26)(x+1)^{26}$

TABLE 3. Isomorphism classes in the family $G_f(D)$ for $f = x^2(x+1) \in \mathbb{F}_3[x]$

Proof. By the Chinese remainder theorem

$$R = \mathbb{F}_q[x]/(f_1 f_2) \cong \mathbb{F}_q[x]/f_1 \times \mathbb{F}_2[x]/f_2.$$

Under this isomorphism and by the definition of D_1, we can see that

$$(x_1, y_1), (x_2, y_2) \in \mathbb{F}_q[x]/f_1 \times \mathbb{F}_q[x]/f_2,$$

are adjacent in $G_f(D_1)$ if and only $y_1 - y_2 \in (\mathbb{F}_q[x]/f_2)^\times$. We then see that $G_f(D_1)$ is isomorphic to the wreath product $G_{f_2}(\{1\}) * E_{m_1}$ where $m_1 = |f_1|$. We remark that we can also get this isomorphism by observing that $f_2 R$ is a homogeneous set in $G_f(D_1)$ and the required isomorphism follows from [8, Theorem 5.5].

By the result from the previous section, we further have

$$G_{f_2}(\{1\}) \cong G_{\mathrm{rad}(f_2)}(\{1\}) * E_{m_2},$$

where $m_2 = |f_2|/|rad(f_2)|$. Since the wreath product is associative, we have

$$G_f(D_1) \cong G_{\mathrm{rad}(f_2)}(\{1\}) * E_m,$$

where $m = \frac{|f|}{|\mathrm{rad}(f_2)|}$. An identical argument and the fact that $\mathrm{rad}(f_1), \mathrm{rad}(f_2)$ have the same factorization type show that $G_f(D_2)$ is isomorphic to $G_{\mathrm{rad}(f_1)}(\{1\}) * E_m$ as well. We conclude that $G_f(D_1) \cong G_f(D_2)$. $\square$

Remark 4.10. We thank Professor Ki-Bong Nam for some discussion which leads to the statement for Proposition 4.9. Specifically, Professor Nam suggested us to study generalized Euler numbers which we now recall. Let A be a PID, $n \in A$ and m is a divisor of n. The generalized Euler number $\varphi_m(n)$ is the number of elements in the following set

$$U_m(n) = \{a \in A/n \mid \gcd(a, m) = 1\}.$$

When $m = n$, $|U_m(n)|$ is precisely the Euler totient function of n. In the setting of Proposition 4.9, $S_{D_1} = U_{f_2}(f_1 f_2)$ where S_{D_1} is the generating set of the gcd-graph $G_f(D_1)$.

We have a simple corollary which is a by-product of the proof for Proposition 4.9.

Corollary 4.11. *Let f be a polynomial. Suppose that f has distinct irreducible factors f_1, f_2 of the same degree. Let a_1, a_2 be two positive integers such that $f_1^{a_1} \mid\mid f$ and $f_2^{a_2} \mid\mid f$. Let*

$$D_1 = \mathrm{Div}(f_1^{a_1}) \cup \{f_1^{a_1}\},$$

and

$$D_2 = \mathrm{Div}(f_2^{a_2}) \cup \{f_2^{a_2}\}.$$

Then $G_f(D_1) \cong G_f(D_2)$.

Proof. For $i \in \{1, 2\}$, let $h_i = f/f_i^{a_i}$. By the same proof as Proposition 4.9 applied to $f = f_i^{a_i} h_i$ we know that

$$G_f(D_i) \cong G_{\mathrm{rad}(h_i)}(\{1\}) * E_{|f/\mathrm{rad}(h_i)|}.$$

By definition, we know that $\mathrm{rad}(h_i)$ and $\mathrm{rad}(h_2)$ have the same factorization type. Since these polynomials are monic, we conclude that $f/\mathrm{rad}(h_1) \cong f/\mathrm{rad}(f_2)$. Therefore, $G_f(D_1) \cong G_f(D_2)$. ☐

For the second phenomenon described in Example 4.8, we found the following.

Proposition 4.12. *Let f_1, f_2 be two irreducible polynomials of the same degree. Let $n \geq 2$ be a positive integer, $f = f_1^n f_2$,*

$$D_1 = \{1, f_2, f_1^2, \ldots, f_1^n\},$$

and

$$D_2 = \{1, f_2, f_1 f_2\}.$$

Then $G_f(D_1)$ and $G_f(D_2)$ are isomorphic.

Proof. By [8, Theorem 5.5], the ideal I_{f_1} generated by f_1 is a homogeneous set in $G_f(D_1)$ as well as $G_f(D_2)$. Furthermore

$$G_f(D_1) \cong G_{f_1}(1) * G_{f_1^{n-1} f_2}(\{f_1, f_1^2, \ldots, f_1^{n-1}\}).$$

Similarly, we also have

$$G_f(D_2) \cong G_{f_1}(1) * G_{f_1^{n-1} f_2}(\{f_2\}).$$

Therefore, to complete the proof, we only need to show that

$$G_{f_1^{n-1} f_2}(\{f_1, f_1^2, \ldots, f_1^{n-1}\}) \cong G_{f_1^{n-1} f_2}(\{f_2\}).$$

We observe that $G_{f_1^{n-1}f_2}(\{f_1, f_1^2, \ldots, f_1^{n-1}\})$ is isomorphic to $|f_1|$ disjoint copies of $G_{f_1^{n-2}f_2}(\{1, f_1, \ldots, f_1^{n-2}\})$ and $G_{f_1^{n-1}f_2}(\{f_2\})$ is isomorphic to $|f_2|$ disjoint copies of $G_{f_1^{n-1}}(\{1\})$. By Theorem 3.5 and the proof of Proposition 4.9 we have

$$G_{f_1^{n-2}f_2}(\{1, f_1, \ldots, f_1^{n-2}\}) \cong K_{|f_1|} * E_{|f_1|^{n-2}} \cong G_{f_1^{n-1}}(\{1\}).$$

Summarizing all these isomorphisms, we conclude that $G_f(D_1)$ and $G_f(D_2)$ are isomorphic. □

Remark 4.13. A by-product of the above proof is that if f_1 and f_2 are two irreducible polynomials of the same degree then for each $m \geq 1$

$$G_{f_1^m f_2}(\{1, f_1, \ldots, f_1^m\}) \cong K_{|f_1|} * E_{|f_1|^m} \cong G_{f_1^{m+1}}(\{1\}).$$

This isomorphism provides a simple construction of two isomorphic gcd-graphs with different moduli.

Remark 4.14. A conjecture of So [13, Conjecture 7.3] says that if n is an integer and D_1, D_2 are two distinct subsets of $\mathrm{Div}(n)$ then the two gcd-graphs over $\mathbb{Z}$, $G_n(D_1)$ are $G_n(D_2)$ are not isomorphic.

As we have shown, this conjecture is wrong if we consider gcd-graph over $\mathbb{F}_q[x]$. We remark, however, that, our constructions of isomorphic gcd-graphs of the form $G_f(D)$ where f is fixed are based on a crucial fact that f has two irreducible factors of the same order. This could happen over $\mathbb{F}_q[x]$ but not over $\mathbb{Z}$. As a result, we wonder whether So's conjecture still holds for the family $G_f(D)$ under the additional assumption that irreducible factors of f have different order.

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