ON DIVISOR GRAPHS

JONATHAN L. MERZEL, JÁN MINÁČ, TUNG T. NGUYEN, NGUYễN DUY TÂN

ABSTRACT. For each positive integer n, we define the divisor graph D_n whose vertex set is the set of divisors of n, and two vertices are adjacent if one is a divisor of the other. This type of graph is a special case of graphs associated with a partial order, which have been widely studied in the literature. In this work, we determine various graph-theoretic invariants of divisor graphs, such as their clique and independence numbers, and their planarity. We also discuss various spectral properties that are discovered by our numerical experiments. Our argument is number-theoretic in nature, making it accessible to a wide range of readers.

1. INTRODUCTION

The divisor graph G_n on n nodes is the graph whose vertices are $\{1, 2, ..., n\}$ and edges are (a, b) where either $a \mid b$ or $b \mid a$. Due to its arithmetical nature, these graphs have been widely studied in the literature. For example, there exists an extensive line of work investigating the longest simple path on G_n (see [5, 8, 11, 12]). Additionally, the determination of the independent and clique numbers of G_n has also attracted significant interest (see, for example, [1, 2, 7]).

While studying the gcd-graphs in [9], we naturally rediscovered G_n . Specifically, in our attempts to understand certain induced structures on the gcd-graphs and to prove several statements about their properties, we realized the importance of understanding the connection scheme between the given divisor subset. Therefore, quite naturally, for our purposes, we restrict ourselves to the induced subgraph D_n of G_n on the set of divisors of n. We note that the definition of D_n is also introduced in [13, Definition 19]. To start our discussion, let us provide the following formal definition of D_n .

Definition 1.1. The divisor graph D_n is the graph with the following data

- (1) The vertice set of D_n is the set of all divisors of n.
- (2) Two vertices a, b are adjacent if $a \mid b$ or $b \mid a$.

The graph D_n is a special case of compatibility graphs; i.e. graphs which are associated with an order (we will explain this interpretation in more detail in Section 2.1). Therefore, D_n is somewhat more structured than G_n . For example, a classical theorem of Dilworth shows that compatibility graphs are perfect (see [4, 6]). In particular, D_n is a perfect graph. Furthermore, we can explicitly calculate some of its variants, such as the clique number (and hence the chromatic number since D_n is perfect), the independence number, and their planarity. More surprisingly, we found that the spectrum of D_n has several interesting and mysterious properties. Although some of them can be explained by number-theoretic or set-theoretic arguments, others are somewhat more mysterious. 1.1. **Outline.** In Section 2, we discuss various graph-theoretic properties of D_n . Most arguments in this section are quite elementary, which makes it accessible to a wide range of readers. Our main discoveries are in Section 3 where we discuss some variational properties of the spectrum of D_n when n varies. In particular, we find that the case where n has at most two prime factors has some quite surprising patterns.

2. Graph theoretic properties of D_n

2.1. **Isomorphism classes and equivalent interpretations.** For each natural number *n*, we can factor *n* into a product of prime factors

$$n=p_1^{a_1}p_2^{a_2}\dots p_d^{a_d}$$

If we require further that $1 \le a_1 \le a_2 \le \ldots \le a_d$ then this factorization is unique.

Definition 2.1. We call (a_1, a_2, \ldots, a_d) the factorization type of *n*. We will write

$$F(n) = (a_1, a_2, \ldots, a_d).$$

We also define the degree deg(*n*) of *n* to be deg(*n*) = $\sum_{i=1}^{d} a_i$.

We now describe an equivalent definition of D_n . Let S be the following set

$$S = \{ (\alpha_1, \alpha_2, \dots, \alpha_d) \mid 0 \le \alpha_i \le a_i \quad \forall 1 \le i \le d \}$$

Definition 2.2. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and $\boldsymbol{\beta} = (\beta_1, \beta, \dots, \beta_d)$ be two vectors in \mathbb{R}^d . We say that $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if and only if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq d$.

We can see that *S* together with this order is a partially ordered set. Let Div(n) be the set of divisors of *n*. For each $d \in \text{Div}(n)$, *d* has a unique factorization of the form $\prod_{i=1}^{d} p_i^{\alpha_i}$ where $0 \leq \alpha_i \leq a_i$. As a result, we have a canonical a map $\Phi : \text{Div}(n) \to S$ defined by $\Phi(d) = (\alpha_1, \ldots, \alpha_d)$. By the fundamental theorem of arithmetic, Φ is a bijection. Furthermore, $d_1|d_2$ if and only if $\Phi(d_1) \leq \Phi(d_2)$. We conclude that D_n is ismorphic to the compatibility graph associated with (S, \leq) . In particular, the isomorphism class of D_n only depends on the factorization type of *n*. While this interpretation of D_n may seem almost tautological, it proves useful—especially when constructing the graph using the Python library networkx, as it allows us to avoid explicit factorization of large numbers.

2.2. Degrees and edges in D_n .

Proposition 2.3. If the factorization type of *n* is (a_1, \dots, a_d) then the graph D_n has $v := \prod_{i=1}^d (a_i + 1)$ vertices and $e = v \left(\frac{(a_1+2)\cdots(a_d+2)}{2^d} - 1 \right)$ edges.

Proof. The vertex count is just the standard result on the number of divisors of an integer. To count edges, we orient the edges by considering an edge joining vertices b, c to originate at b and terminate at c if b|c. Then, if for $i = 1, \dots, d$ we have $0 \le x_i \le a_i$, the number of edges originating at $b = \prod_{i=1}^{d} p_i^{x_i}$ is $\prod_{i=1}^{d} (a_i + 1 - x_i) - 1$. (We subtract 1 as we do not include a

loop at *b*.) So the total number of edges is given by

$$\sum_{x_1=0}^{a_1} \cdots \sum_{x_d=0}^{a_d} \left(\prod_{i=1}^d (a_i + 1 - x_i) - 1 \right)$$

Reindexing by setting $w_i = a_i + 1 - x_i$, we can rewrite this as

$$\sum_{w_{1}=1}^{a_{1}+1} \cdots \sum_{w_{d}=1}^{a_{d}+1} \left(\prod_{i=1}^{d} w_{i} - 1 \right) = \sum_{w_{1}=1}^{a_{1}+1} \cdots \sum_{w_{d}=1}^{a_{d}+1} \prod_{i=1}^{d} w_{i} - v$$

$$= \left(\sum_{w_{1}=1}^{a_{1}+1} w_{1} \right) \cdots \left(\sum_{w_{d}=1}^{a_{d}+1} w_{d} \right) - v$$

$$= \frac{(a_{1}+1)(a_{1}+2)}{2} \cdots \frac{(a_{d}+1)(a_{d}+2)}{2} - v =$$

$$= v \left(\frac{(a_{1}+2)\cdots(a_{d}+2)}{2^{d}} - 1 \right)$$

2.2.1. Vertices of Minimal Degree.

Definition 2.4. Let \mathbf{e}_i be the unit vector with 0 in position $j, 1 \le j \ (\ne i) \le d$ and 1 in position i. Vectors $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{x} \pm \mathbf{e}_i$ for some i will be called <u>close</u>. A sequence $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a <u>chain</u> iff \mathbf{x}_i and \mathbf{x}_{i+1} are close, $1 \le i \le n-1$.

Definition 2.5. A vector $\mathbf{x} = (x_1, \dots, x_d)$ is called <u>extremal in coordinate *i*</u> if $x_i = 0$ or $x_i = a_i$, and is called <u>extremal</u> if it is extremal in all coordinate positions. The term nonextremal applies otherwise in both situations.

Remark 2.6. The degree of a vertex $\mathbf{x} = (x_1, \dots, x_d)$ is easily seen to be given by

$$f(\mathbf{x}) := \prod_{i=1}^{d} (a_i + 1 - x_i) + \prod_{i=1}^{d} (x_i + 1) - 2.$$

For $1 \le i \le d$ let $\Delta_i(\mathbf{x}) = f(\mathbf{x} + \mathbf{e}_i) - f(\mathbf{x}) = \prod_{j \ne i} (x_j + 1) - \prod_{j \ne i} (a_j + 1 - x_j)$, and note that $\Delta_i(\mathbf{x})$ does not depend on the value of x_i . It follows that if $\Delta_i(\mathbf{x}) < 0$,

 $\mathbf{x} - x_i \mathbf{e}_i, \mathbf{x} - (x_i - 1) \mathbf{e}_i, \cdots, \mathbf{x} + (a_i - x_i) \mathbf{e}_i)$

is a chain through **x**, joining the vectors $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d)$ and $(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)$ extremal in coordinate *i*, with strictly decreasing degrees. Similarly if $\Delta_i(\mathbf{x}) > 0$, we get such a chain with strictly increasing degrees, and if $\Delta_i(\mathbf{x}) = 0$, the chain maintains constant degree.

Proposition 2.7. A vertex of minimal degree can always be found among the extremal vertices. In fact, any vertex of minimal degree satisfies the "stability condition" $\Delta_i(\mathbf{x}) = 0$ at every nonextremal coordinate, and admits a chain from \mathbf{x} to an extremal vertex of minimal degree where every member of this chain has minimal degree.

Proof. Let **x** be any vertex with minimal degree. The argument in the above remark implies that **x**, if not alreadly extremal, must satisfy $\Delta_i(\mathbf{x}) = 0$ at every nonextremal coordinate (else we can move to a close vertex of lower degree); choosing a nonextremal coordinate position

i, **x** occurs in a chain with constant degree leading to a vertex differing only in coordinate *i* where it is now extremal. Since all vertices in this chain are still of minimal degree, the stability condition is still intact. Repeat with any remaining nonextremal coordinate positions until arriving at an extremal vertex. The construction ensures that the degree is not changed in this process.

Proposition 2.8. An extremal vertex **x** has minimal degree is and only if $\prod_{i \in S} (a_i + 1) + \prod_{j \in T} (a_j + 1)$ is as small as possible among extremal vertices, where $S = \{i \mid x_i = a_i\}$ and $T = \{j \mid x_j = 0\}$.

Proof. This follows simply from the formula for $f(\mathbf{x})$, which for an extremal vertex \mathbf{x} computes to $\prod_{i \in S} (a_i + 1) + \prod_{j \in T} (a_j + 1) - 2$.

2.3. **Connectedness and bipartite property.** The graph D_n is always connected since 1 and n are adjacent to all other vertices. Furthermore, we have the following property.

Proposition 2.9. Let *a*, *b* be two vertices in D_n . Let d(a, b) be the distance between *a* and *b*. Then

$$d(a,b) = \begin{cases} 1 & \text{if } (a,b) \in E(D_n) \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If *a* and *b* are adjacent then d(a, b) = 1. Otherwise, we would have the path $a \to 1 \to b$ and hence d(a, b) = 2.

Proposition 2.10. Let G be the induced graph on $D_n \setminus \{1, n\}$. Then, G is connected unless n = pq where p, q are distinct prime numbers; i.e, the factorization type of n is (1, 1).

Proof. If n = pq where p, q are distinct prime then we can see that the induced subgraph on $\{p, q\}$ is the empty graph and hence it is not connected. Conversely, suppose that the factorization type of n is not (1, 1). If n has exactly one prime divisor then the induced subgraph on $D_n \setminus \{1, n\}$ is a complete graph and therefore it is connected connected. Let us now assume that n has at most two prime factors. Since a vertex is adjacent to all its divisor, it is sufficient to show that if p, q are distinct prime divisor of n then p, q are connected. This is clear since we have the path $p \to pq \to p$.

Proposition 2.11. D_n is bipartite if and only F(n) = (1); *i.e.*, *n* is a prime number.

Proof. If *n* is not a prime number then there exists 1 < d < n such that $d \mid n$. In this case $\{1, d, n\}$ forms a triangle in D_n . As a result, D_n cannot be bipartite.

2.4. Clique and independent numbers of D_n .

Proposition 2.12. Suppose that $F(n) = (a_1, a_2, ..., a_d)$. Let X be a maximal clique in D_n . Then

$$|X| = 1 + \sum_{i=1}^{d} a_i = 1 + \deg(n).$$

Proof. We will prove by induction on *d* that there is a clique of size $1 + \deg(n)$. For d = 1, $n = p_1^{a_1}$ is a prime power and hence D_n is the complete graph on $a_1 + 1$ nodes. In this case, $X = V(D_n)$ is a clique of size $1 + \deg(n)$. Let us assume that the statement is true for *d*. Let us

show that it is true for d + 1 as well. Let us write $n = \prod_{i=1}^{d+1} p_i^{a_i}$. Let $X = \{h_1, h_2, \dots, h_{1+\sum_{i=1}^d a_d}\}$ be a clique of size $1 + \sum_{i=1}^d a_i$ in $D_{\prod_{i=1}^d p_i^{a_i}}$. We can then check that $X' = \{1, p_{d+1}, \dots, p_{d+1}^{a_d-1}\} \cup p_{d+1}^{a_d} X$ is a clique of size $1 + \sum_{i=1}^{d+1} a_i$ in $D_{\prod_{i=1}^{d+1} p_i^{a_i}}$.

Conversely, we show that a clique in D_n has at most $1 + \deg(n)$ elements, In fact, let $X = \{h_1, h_2, \ldots, h_q\}$ where $h_1 < h_2 < \ldots < h_q$. We claim that $|X| \le 1 + \deg(n)$. Since X is a clique and $h_i < h_{i+1}$, $h_i|h_{i+1}$ for each $0 \le i \le q-1$. This shows that $0 \le \deg(h_1) < \deg(h_2) \ldots < \deg(h_q) \le \deg(n)$. Consequently, we conclude that $|X| = q \le 1 + \deg(n)$. \Box

For the independence number of D_n , we have the following result.

Proposition 2.13. (See [3, Theorem 1]) Let S be a subset of D_n consisting of vertices of the same degree $\lfloor \frac{\deg(n)}{2} \rfloor$. Then S is a maximal independence set in D_n . In particular, the independence number of D_n is precisely |S|.

2.4.1. *Perfectness and coloring of* D_n . Since D_n is perfect, the chromatic number of D_n is equal to its clique number; i.e, $\chi(D_n) = 1 + \deg(n)$. In fact, we can provide an explicit coloring on D_n with $1 + \deg(n)$ colors. Specifically, we can define a coloring map $c : D_n \rightarrow \{0, 1, \dots, \deg(n)\}$ by the rule $c(m) = \deg(m)$. Clearly, two numbers with the same degree cannot be adjacent, c is a coloring map.

2.5. **Planarity of** D_n . In this section, we classify all D_n which are planar. By exploring D_n with some small values of n, we can see that $F(n) = (a_1, a_2, ..., a_d)$ with $d \le 2$ and $\sum_{i=1}^d a_i \le 3$ then D_n is planar. See Fig. 1 for a planar embedding when F(n) = (1, 2) and Fig. 2 for a planar embedding when F(n) = (3).



FIGURE 1. A planar embedding of D_{45}

FIGURE 2. A planar embedding of D_{27} .

The following theorem completely classify n such that D_n is planar.

Proposition 2.14. *The graph* D_n *is planar if and only if one of the following holds:*

(1) n = p^r, p prime, 0 ≤ r ≤ 3
(2) n = pq, p, q prime
(3) n = p²q, p, q prime

Proof. We make use of the fact that a graph cannot be planar if it contains a subgraph isomorphic to $K_{3,3}$ or to a subdivision of K_5 . Since if D_n is planar and $m|n, D_m$ (a subgraph, induced or not) is also planar, and if D_n is not planar and $n|m, D_m$ is also not planar, it suffices to check the following cases (where p, q, r are distinct primes)

- (1) D_{p^3} is planar (see Fig. 2) and D_{p^4} is not. This is true since D_{p^r} is the complete graph K_{r+1} which is planar iff $r + 1 \le 4$.
- (2) D_{pq} and D_{p^2q} are planar (see Fig. 1).
- (3) $D_{p^2q^2}$ is not planar. For it contains a subgraph which is a subdivision of K_5 .



(4) D_{p^3q} is not planar. It contains the following subgraph which isomorphic to $K_{3,3}$.



(5) D_{pqr} is not planar.



It fails the known necessary planarity condition $e \le 3v - 6$. Alternatively, we note that it contains a subgraph which is a subdivision of K_5 .

Remark 2.15. By some elementary calculations, we can in fact see that D_n is planar if and only if it satisfies condition $e \le 3v - 6$.

3. Spectrum of D_n

In this section, we study the spectrum of D_n .

3.1. Variation of the spectra of D_n . For each n, let f_n be the characteristics polynomial of D_n .

Proposition 3.1. Let *n* be a natural number. Let *p*, *q* be prime numbers such that gcd(n, pq) = 1. Then $f_n \mid f_{npq}$. Furthermore, if *n* is squarefree, then $f_n^2 \mid f_{npq}$.

Proof. Suppose first that p, q are distinct primes and that gcd(n, pq) = 1. Let $1 = a_1 < \cdots < a_k = n$ be the divisors of n, and let B be the adjacency matrix of D_n with respect to this order. Let the $k \times k$ matrix C be defined by $C_{ij} = 1$ if $a_i | a_j$ and 0 otherwise. (Then $B = C + C^T - 2I_k$.) Now list divisors of npq in the order $a_1, \cdots, a_k, pa_1, \cdots, pa_k, qa_1, \cdots, qa_k, pqa_1, \cdots, pqa_k$. With respect to this order, the adjacency matrix for D_{npq} is

$$M = \begin{bmatrix} B & C & C & C \\ C^{T} & B & 0 & C \\ C^{T} & 0 & B & C \\ C^{T} & C^{T} & C^{T} & B \end{bmatrix}$$

Write I_k for the $k \times k$ identity matrix and $\mathbf{0}_k$ for an $k \times k$ matrix of 0's. The characteristic polynomial of *M* can be analyzed as follows, where for the third equality, we have subtracted

the second row of blocks from the third:

$$f_{npq} = \det(M - \lambda I_{n+4}) = \det \begin{bmatrix} B - \lambda I_k & C & C & C \\ C^T & B - \lambda I_k & \mathbf{0}_k & C \\ C^T & \mathbf{0}_k & B - \lambda I_k & C \\ C^T & C^T & C^T & B - \lambda I_k \end{bmatrix}$$
$$= \det \begin{bmatrix} B - \lambda I_k & C & C & C \\ C^T & B - \lambda I_k & \mathbf{0}_k & C \\ \mathbf{0}_k & \lambda I_k - B & B - \lambda I_k & \mathbf{0}_k \\ C^T & C^T & C^T & B - \lambda I_k \end{bmatrix}$$
$$= \det \begin{bmatrix} I_k & \mathbf{0}_k & \mathbf{0}_k & \mathbf{0}_k \\ \mathbf{0}_k & I_k & \mathbf{0}_k & \mathbf{0}_k \\ \mathbf{0}_k & \mathbf{0}_k & \mathbf{0}_k & I_k \end{bmatrix} \det \begin{bmatrix} B - \lambda I_k & C & C & C \\ C^T & B - \lambda I_k & \mathbf{0}_k & \mathbf{0}_k \\ \mathbf{0}_k & -I_k & I_k & \mathbf{0}_k \\ C^T & C^T & C^T & C^T & B - \lambda I_k \end{bmatrix}$$
$$= \det(B - \lambda I_k) \cdot (a \text{ polynomial in } \lambda) = f_n \cdot (a \text{ polynomial in } \lambda).$$

This shows $f_n \mid f_{npq}$.

For the second assertion, suppose that we can write $n = n_0 r$, where r is a prime number, and $gcd(r, n_0) = 1$. (This condition is satisfied, for example, when n > 1 is squarefree.)

Let $a_1 = 1 < a_2 < \cdots < a_k$ be the divisors of n_0 . Let B_0 be the adjacency matrix of D_{n_0} . Let *C* be the matrix defined by

$$C_{ij}=1\iff a_i\mid a_j.$$

List the divisors of $n_0 r p q$ in the order:

$$\underbrace{a_1, \dots, a_k}_{(1)}, \underbrace{ra_1, \dots, ra_k}_{(2)}, \underbrace{pa_1, \dots, pa_k}_{(3)}, \underbrace{pra_1, \dots, pra_k}_{(4)}, \underbrace{qa_1, \dots, qa_k}_{(5)}, \underbrace{qra_1, \dots, qra_k}_{(6)}, \underbrace{pqa_1, \dots, pqa_k}_{(7)}, \underbrace{pqra_1, \dots, pqra_k}_{(8)}, \underbrace{pqra_1, \dots, pqra_k}_{(8$$

We note that

$$C_{ij} = 1 \iff a_i \mid a_j \quad \text{or} \quad ra_i \mid ra_j.$$

Then the adjacency matrix for $D_{n_0 r p q}$ is of the form:

$$M = \begin{bmatrix} B_0 & C & C & C & C & C & C & C \\ C^T & B_0 & 0 & C & 0 & C & 0 & C \\ C^T & 0 & B_0 & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 & 0 & 0 & 0 & C \\ C^T & 0 & 0 & 0 & B_0 & C & C & C \\ C^T & C^T & 0 & 0 & C^T & B_0 & 0 & C \\ C^T & C^T & 0 & C^T & 0 & C^T & 0 & B_0 & C \\ C^T & 0 & C^T & 0 & C^T & 0 & B_0 & C \\ C^T & C^T & C^T & C^T & C^T & C^T & B_0 \end{bmatrix} = \begin{bmatrix} B & A & A & A \\ A^T & B & 0 & A \\ A^T & 0 & B & A \\ A^T & A^T & A^T & B \end{bmatrix},$$

where

$$B = \begin{bmatrix} B_0 & C \\ C^T & B_0 \end{bmatrix}, \quad A = \begin{bmatrix} C & C \\ 0 & C \end{bmatrix}.$$

We have

$$|M - \lambda I_N| = \begin{vmatrix} B_0 - \lambda I & C & C & C & C & C & C & C \\ C^T & B_0 - \lambda I & 0 & C & 0 & 0 & C & C \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 - \lambda I & 0 & 0 & 0 & C \\ C^T & 0 & 0 & 0 & B_0 - \lambda I & C & C & C \\ C^T & C^T & 0 & C^T & 0 & C^T & 0 & B_0 - \lambda I & C \\ C^T & B_0 - \lambda I \end{vmatrix}$$
$$= \pm \begin{vmatrix} B_0 - \lambda I & C & C & C & C & C \\ 0 & B_0 - \lambda I & C & C & C & C & C & C \\ 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 & 0 & C & -C & 0 \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & C^T & C^T & C^T & B_0 - \lambda I & 0 & 0 & 0 & C \\ C^T & 0 & B_0 - \lambda I & -C & B_0 - \lambda I & C & 0 & 0 \\ 0 & 0 & -B_0 + \lambda I & -C & B_0 - \lambda I & C & 0 & 0 \\ 0 & 0 & -C^T & -B_0 + \lambda I & C^T & B_0 - \lambda I & 0 & 0 \\ 0 & 0 & -C^T & C^T & C^T & C^T & C^T & B_0 - \lambda I & 0 \end{vmatrix}$$

Here, $I = I_k$ and we the following (block) row operations:

$$(2): \quad R_2 \leftarrow R_2 - R_3$$

$$(5): \quad R_5 \leftarrow R_5 - R_3$$

$$(6): \quad R_6 \leftarrow R_6 - R_4$$

$$(7): \quad R_7 \leftarrow R_6 - R_7.$$

Thus

In the above product of determinants, the first determinant is f_n . For the second determinant, which we denote temporarily as *S*, we move the block rows *R*2 and *R*7 to the top, we obtain

$$S = \pm \begin{vmatrix} 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 & 0 & C & -C & 0 \\ 0 & C^T & -C^T & 0 & 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 \\ B_0 - \lambda I & C & C & C & C & C & C \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 - \lambda I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & I & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & I & 0 & 0 \\ C^T & B_0 - \lambda I \end{vmatrix}$$

Now in the above determinant, we move the block columns C_2 , C_6 , C_3 and C_7 (in this order) to the left, we imply that

 $= \pm f_n \times (a \text{ polynomial in } \lambda).$

Hence f_{npq} is divisible by f_n^2 .

3.2. **Special eigenvalues.** Let λ be a real number. Then λ is an eigenvalue of D_n if there is a vector $(v_d)_{d|n}$ such that

$$\sum_{(d,m)\in E(D_n)}v_m=\lambda v_d.$$

The case $\lambda = -2$ is special because we can rewrite this as

(3.1)
$$\sum_{m|d} v_m + \sum_{d|m|n} v_m = 0.$$

We will use this fact to construct an eigenvector associated with $\lambda = -2$ under some conditions on *n*.

Let μ be the Moebius function. This function has the property that for each positive integer m

$$\sum_{d|m} \mu(m) = egin{cases} 1 & ext{if } m = 1 \ 0 & ext{else.} \end{cases}$$

Proposition 3.2. Suppose that $\mu(n) = -1$. Let $v = (v_d)_{d|n}$ be the vector defined by

$$v_d = egin{cases} 0 & \textit{if } d \in \{1,n\} \ \mu(d) & \textit{else.} \end{cases}$$

Then v is an eigenvector associated with the eigenvalue $\lambda = -2$.

Proof. We verify that Eq. (3.1) holds for all $d \mid n$. If $d \notin \{1, n\}$ then we have

$$\sum_{m|d} v_m + \sum_{d|m|n} v_m = \sum_{m|d} \mu(m) - \mu(1) + \mu(d) \sum_{m|\frac{n}{d}} \mu(m) - \mu(n) = 0$$

If d = 1 then Eq. (3.1) becomes

$$v_1 + \sum_{m|n} v_m = \sum_{d|n} \mu(d) - \mu(1) - \mu(n) = 0.$$

Similarly, we can check that Eq. (3.1) holds for d = n.

Lemma 3.3. Let X be a clique set in a connected graph G. Suppose further that both X and $G \setminus X$ are homogeneous sets in G. Then -1 is an eigenvalue of G with multiplicity at least |X| - 1.

Proof. By definition, *G* is a joined union of the induced graph on *X* and $G \setminus X$. The conclusion then follows from the proof of [10, Proposition 3.1]. More specifically, the adjacency matrix of *G* is of the form

$$A = \begin{pmatrix} A_X & J \\ J & A_{G \setminus X} \end{pmatrix}.$$

Let $v = (v_1, \ldots, v_{|X|})$ be a vector such that $\sum_{i=1}^{|X|} v_i = 0$. Then, $w = (v_1, \ldots, v_{|X|}, 0, \ldots, 0)$ is an eigenvector for A_G associated with the eigenvalue $\lambda = -1$.

Corollary 3.4. -1 *is an eigenvalue of* D_n *for every* $n \ge 2$.

Proof. $X = \{1, n\}$ sastifies the conditions of Lemma 3.3.

3.3. The case where *n* has at most two prime factors.

Proposition 3.5. Let M_a be an adjacency matrix for D_{pq^a} where p, q are distinct primes, $a \ge 0$. Then $\det(M_a) = \det(M_{a+6})$.

Proof. We note that $det(M_5) = 1$, and explicitly that, with respect to the vertex ordering $1, q, \dots, q^5, p, pq, \dots, pq^5$

$M_5 =$	0	1	1	1	1	1	1	1	1	1	1	1
	1	0	1	1	1	1	0	1	1	1	1	1
	1	1	0	1	1	1	0	0	1	1	1	1
	1	1	1	0	1	1	0	0	0	1	1	1
	1	1	1	1	0	1	0	0	0	0	1	1
	1	1	1	1	1	0	0	0	0	0	0	1
	1	0	0	0	0	0	0	1	1	1	1	1
	1	1	0	0	0	0	1	0	1	1	1	1
	1	1	1	0	0	0	1	1	0	1	1	1
	1	1	1	1	0	0	1	1	1	0	1	1
	1	1	1	1	1	0	1	1	1	1	0	1
	1	1	1	1	1	1	1	1	1	1	1	0

and

$$M_{5}^{-1} = \begin{bmatrix} -2 & 1 & 2 & 1 & -1 & -2 & -2 & -1 & 1 & 2 & 1 & -1 \\ 1 & -2 & -1 & 0 & 1 & 1 & 2 & 0 & -1 & -1 & 0 & 1 \\ 2 & -1 & -4 & -1 & 2 & 3 & 3 & 2 & -2 & -3 & -1 & 2 \\ 1 & 0 & -1 & -2 & 1 & 2 & 1 & 1 & 0 & -2 & -1 & 1 \\ -1 & 1 & 2 & 1 & -2 & -1 & -2 & -1 & 1 & 2 & 0 & -1 \\ -2 & 1 & 3 & 2 & -1 & -4 & -3 & -2 & 1 & 3 & 2 & -2 \\ -2 & 2 & 3 & 1 & -2 & -3 & -4 & -1 & 2 & 3 & 1 & -2 \\ -1 & 0 & 2 & 1 & -1 & -2 & -1 & -2 & 1 & 2 & 1 & -1 \\ 1 & -1 & -2 & 0 & 1 & 1 & 2 & 1 & -2 & -1 & 0 & 1 \\ 2 & -1 & -3 & -2 & 2 & 3 & 3 & 2 & -1 & -4 & -1 & 2 \\ 1 & 0 & -1 & -1 & 0 & 2 & 1 & 1 & 0 & -1 & -2 & 1 \\ -1 & 1 & 2 & 1 & -1 & -2 & -2 & -1 & 1 & 2 & 1 & -2 \end{bmatrix}$$

Order the vertices of M_a as $1, q, \dots, q^a, p, pq, \dots, pq^a$, and the vertices of M_{a+6} as $1, q, \dots, q^a, p, pq, \dots, pq^a, q^{a+1}, \dots, q^{a+6}, pq^{a+1}, \dots pq^{a+6}$. Now if we write $\mathbf{1}_{r,s}$ for an $r \times s$ matrix of 1's and $\mathbf{0}_{r,s}$ for an $r \times s$ matrix of 0's, we have

$$M_{a+6} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A = M_a$, $B = \begin{bmatrix} \mathbf{1}_{a+1,6} & \mathbf{1}_{a+1,6} \\ \mathbf{0}_{a+1,6} & \mathbf{1}_{a+1,6} \end{bmatrix}$, $C = B^T$, and $D = M_5$. Since D is invertible, $\det(M_{a+6}) = \det(D) \det(A - BD^{-1}C).$

But det(*D*) = 1, and using the explicit computation of $D^{-1} = M_5^{-1}$ above, $BD^{-1}C = \mathbf{0}_{2a+2,2a+2}$. (We can check this by verifying $\mathbf{1}_{1,12}D^{-1}\mathbf{1}_{12,1} = [\mathbf{0}_{1,6} \mathbf{1}_{1,6}]D^{-1}\mathbf{1}_{12,1} = \mathbf{1}_{1,12}D^{-1}\begin{bmatrix}\mathbf{0}_{6,1}\\\mathbf{1}_{6,1}\end{bmatrix} = [\mathbf{0}_{1,6} \mathbf{1}_{1,6}]D^{-1}\begin{bmatrix}\mathbf{0}_{6,1}\\\mathbf{1}_{6,1}\end{bmatrix} = 0.$) Therefore det(M_{a+6}) = det(A) = det(M_a).

Proposition 3.6. For distinct primes p, q and $a \ge 0$, D_{pq^a} has eigenvalue 0 if and only if $a \equiv 1$ mod 6.

Proof. After explicitly computing det(M_a) = -1, 0, 3, 5, 4, 1 for a = 0, 1, 2, 3, 4, 5 respectively, this follows immediately from the previous proposition.

Proposition 3.7. If $n = p^a q^b$ and $a \equiv b \equiv 1 \pmod{6}$ then 0 is an eigenvalue of D_n .

Proof. Suppose that

$$n = p^u q^v$$
 with $u \equiv v \equiv 1 \pmod{6}$.

List the divisors of *n* in the order:

$$1, q, q^2, \ldots, q^v, p, pq, pq^2, \ldots, pq^v, \ldots, p^u, p^uq, \ldots, p^uq^v.$$

The adjacency matrix of D_n is of the form:

$$M = \begin{bmatrix} V & U & U & \cdots & U \\ U^{T} & V & U & \cdots & U \\ U^{T} & U^{T} & V & \cdots & U \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U^{T} & U^{T} & U^{T} & \cdots & V \end{bmatrix},$$

where

$$V = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{(v+1) \times (v+1)} \text{ and } U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(v+1) \times (v+1)}$$

Consider the following column matrices of size $(v + 1) \times 1$:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 & \cdots & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}^{T}$$
$$A' = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & -1 & \cdots & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}^{T}.$$

(In *A*, the block $\begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$ is repeated (v-1)/6 times. The matrix *A'* is obtained from A by left-shifting.) Similary, we consider following column matrices of size $(v + 1) \times 1$:

$$B = \begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 & \cdots & -1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}^{T},$$

$$B' = \begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 1 & \cdots & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}^{T}$$

and

$$C = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 & \cdots & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$C' = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 & \cdots & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \end{bmatrix}^{T}.$$

Let *X* be the following column matrix of size $[(u+1)(v+1)] \times 1$:

$$X = \begin{bmatrix} A & A' & B & B' & C & C' & \cdots & A & A' & B & B' & C & C' & A & A' \end{bmatrix}^{T}$$

(In *X*, the block $\begin{bmatrix} A & A' & B & B' & C & C' \end{bmatrix}$ is repeated (u - 1)/6 times.)

We claim MX = 0. In particular, it shows that 0 is an eigenvalue of *M*.

We need to show that for every non-negative integers s, t with $s + t = u \equiv 1 \pmod{6}$, one has

$$\begin{bmatrix} U^T & \cdots & U^T & V & U & \cdots & U \end{bmatrix} X = 0.$$

Here in the first matrix, U^T appears *s* times and *U* appears *t* times.

Note that A + A' + B + B' + C + C' = 0. Hence $U^T(A + A' + B + B' + C + C') = 0 = U(B + B' + C + C' + A + A')$. We only need to consider the following six cases.

Case 1: $s \equiv 0 \pmod{6}$ and $t \equiv 1 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = 0.$$

Let
$$\begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix}$$
. We have
$$s_1 = \sum_{i \ge 2} a_i + \sum_{i \ge 1} a'_i = 1 + (-1) = 0.$$

On the other hand, the difference between the (i + 1)-st row and the *i*th row of [V | U] is the row

 $[0 \cdots 0 1 -1 0 \cdots 0 | 0 \cdots 0 -1 0 \cdots 0].$

Here, the first -1 is in the i + 1-th position and the second -1 is in the (s + 1) + i-th position. We deduce that

$$s_{i+1} - s_i = a_i - a_{i+1} - a'_i = 0.$$

Hence $s_i = 0$, for every *i*.

Case 2: $s \equiv 1 \pmod{6}$ and $t \equiv 0 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & V \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = 0.$$

Let $\begin{bmatrix} U^T & V \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix}$. We have $s_1 = a_1 + \sum_{i \ge 2} a'_i = 0 + 0 = 0.$

On the other hand, the difference between the (i + 1)-st row and the *i*th row of $[U^T | V]$ is the row

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & | & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 \end{bmatrix}.$$

Here, the first 1 is in the i + 1-th position and the second 1 is in the (s + 1) + i-th position. We deduce that

$$s_{i+1} - s_i = a_{i+1} + a'_i - a'_{i+1} = 0.$$

Hence $s_i = 0$, for every *i*.

Using similar arguments as in the previous two cases, we can show that

$$VB + UB' = \begin{bmatrix} s & s & \cdots & s \end{bmatrix}^T = U^TB + VB',$$

where $s = \sum_{i \geq 2} b_i + \sum_{i \geq 1} b'_i = -2 = b_1 + \sum_{i \geq 2} b'_i$, and that

$$VA' + UB = U^T A' + VB = [-2 -2 \cdots -2]^T,$$

 $VB' + UC = U^T B' + VC = 0,$
 $VC + UC' = U^T C + VC' = [2 \ 2 \ \cdots \ 2]^T.$

Case 3: $s \equiv 2 \pmod{6}$ and $t \equiv 5 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & U^T & V & U & U & U & U \end{bmatrix} \begin{bmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{bmatrix} = 0.$$

We have

$$U^{T}(A + A') + VB + U(B' + C + C' + A + A') = -VA' + U^{T}A' + VB - UB$$

= $(U^{T}A' + VB) - (VA' + UB) = 0.$

Case 4: $s \equiv 3 \pmod{6}$ and $t \equiv 4 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & U^T & U^T & V & U & U & U \end{bmatrix} \begin{bmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{bmatrix} = 0.$$

We have

$$U^{T}(A + A' + B) + VB' + U(C + C' + A + A')$$

= $U^{T}(A + A') + VB + U(B' + C + C' + A + A') + U^{T}B + VB' - VB - UB'$
= $U^{T}B + VB' - (VB + UB') = 0.$

Case 5: $s \equiv 4 \pmod{6}$ and $t \equiv 3 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & U^T & U^T & U^T & V & U & U \end{bmatrix} \begin{bmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{bmatrix} = 0.$$

We have

$$U^{T}(A + A' + B + B') + VC + U(C' + A + A')$$

= $U^{T}(A + A' + B) + VB' + U(C + C' + A + A') + U^{T}B' + VC - VB' - UC$
= $U^{T}B' + VC - (VB' + UC) = 0.$

Case 6: $s \equiv 5 \pmod{6}$ and $t \equiv 2 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^{T} & U^{T} & U^{T} & U^{T} & U^{T} & V & U & U \end{bmatrix} \begin{vmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{vmatrix} = 0.$$

We have

$$U^{T}(A + A' + B + B' + C) + VC' + U(A + A')$$

= $U^{T}(A + A' + B + B') + VC + U(C' + A + A') + U^{T}C + VC' - VC - UC'$
= $U^{T}C + VC' - (VC + UC') = 0.$

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SOKA UNIVERSITY OF AMERICA Email address: jmerzel@soka.edu

UNIVERSITY OF WESTERN ONTARIO Email address: minac@uwo.ca

LAKE FOREST COLLEGE Email address: tnguyen@lakeforest.edu

HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY *Email address*: tan.nguyenduy@hust.edu.vn