JOINS OF NORMAL MATRICES, THEIR SPECTRUM, AND APPLICATIONS

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ABSTRACT. Motivated by studies of oscillator networks, we study the spectrum of the join of several normal matrices with constant row sums. We apply our results to compute the characteristic polynomial of the join of several regular graphs. We then use this theorem to study several problems in spectral graph theory. In particular, we provide some simple constructions of Ramanujan graphs and give new proofs for some theorems in the classical book of Cvetković, Rowlinson, and Slobodan.

1. Introduction

Networks of nonlinear oscillators have attracted interest in several scientific domains such as theoretical physics, mathematical biology, power-grid systems, and many more. In our investigation of oscillator networks (see [1, 15, 18]), the networks of several communities joined together often appear and provide some interesting phenomena (see for example [18, Proposition 23] and [15, Proposition 12]). The key idea of these investigations of multi-layer networks is to reduce the study of dynamics on complex networks to simpler networks. In both theory and practice, the adjacency matrix of the original multi-layer network may appear quite complicated. However, using the techniques that we develop here for join graphs, we will see that we can attach to each multi-layer network a reduced matrix which usually has a much smaller size than the original adjacency matrix. Nevertheless, as we show in [16], by considering this reduced matrix, we can obtain good information about the entire complex network. In [18], we study the case where the connection within a community follows a simple rule, namely, each community is a circulant network. In this case, the main theorem in [18], which generalizes the Circulant Diagonalization Theorem (CDT), explicitly describes the spectrum of the joined network. In this article, we generalize this theorem to the case where each community forms a regular graph. This relaxation will allow us to investigate a broader class of networks. In particular, we are able to apply our generalized theorems to study several interesting problems in spectral graph theory.

We remark that since the completion of this article, we have utilized this circle of ideas to study some broadcasting and combining mechanisms on multi-layer networks of oscillators (see [11, 16].) We want to emphasize that the broadcasted solution described Eq. (3.1) has a nice physical interpretation(see [16]). Furthermore, we also extend this line of research further by investigating the question of whether a given graph can be written as a joined union of smaller graphs with a special focus on the case where the graph is highly symmetric (see [4, 17]).

1.1. **Outline.** The structure of this article is as follows. In Section 2, we study some basic spectral properties of normal matrices with constant row sums. In Section 3, we define the joins of these matrices and study their spectral properties. We then apply the main results from this section to give new proofs of several results in [6] for the join of regular graphs. In this way, we provide a new conceptual insight for these statements based on key results in Section 3. Section 4.2 explains a

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simple method to construct Ramanujan graphs using the join construction. We remark here that the construction of Ramanujan graphs is of great interest in network, communication, coding, and number theories. We then discuss the joined union of graphs in Section 4.3. In Section 4.4, we apply the results from the previous sections to study some questions in graph energy. In particular, we propose a question on the relation between the energy of several regular graphs and their joined union. Some notable results in this section are Theorem 4.5 and Theorem 4.6 where we provide some concrete evidence for our question.

Remark 1. We remark that a weaker form of Theorem 3.2 has been discussed previously in [2, 19]. We refer the reader to Remark 3 and Remark 4 regarding our approach. We explain in these remarks why our approach is more flexible than [2, 19] and why we can apply main results in the situations where [2, 19] cannot be applied directly.

2. Normal matrices with constant row sums

We start with a definition.

Definition 1. Let $A = (a_{ij})_{i,j}$ be an $n \times n$ matrix with complex coefficients. We say that A is r_A-row regular if the sum of all entries in each row of A is equal to r_A, namely

$$\forall 1 \leq i \leq n, \sum_{j=1}^{n} a_{ij} = r_A.$$

Similarly, we say that A is c_A-column regular if the sum of all entries in each column of A is equal to c_A.

Remark 2. Some authors use the term "semimagic squares" for matrices that are both r_A-regular and c_A-regular and $r_A = c_A$ (see, for example [13].)

We note that if A is both r_A-row regular and c_A-column regular then $r_A = c_A$ as long as we work with matrices with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (or more generally, over a field of characteristics 0). This can be seen by observing that the sum of all entries in A is equal to both nr_A and nc_A; and therefore $r_A = c_A$. Here is a simple criterion for row and column regularity.

Lemma 2.1. *Let $v = \mathbb{1}_n = (1, 1, \ldots, 1)^t \in \mathbb{C}^n$. Then A is r_A-row regular if and only if v is an eigenvector of A associated with the eigenvalue r_A. Similarly, A is c_A-column regular if and only if v^t is a left eigenvector of A associated with the eigenvalue c_A.*

Proof. Obvious from the definition. □

Definition 2. Let $A \in M_n(\mathbb{C})$ be a matrix of size $n \times n$. We say that A is normal if $AA^* = A^*A$. Here A^* is the conjugate transpose of A.

A special property of normal matrices is that they are always diagonalizable by an orthonormal basis of eigenvectors.

Theorem 2.1. *(see* [10, Theorem 2.5.3]*) Suppose A is a normal matrix. Then its eigenspaces span $\mathbb{C}^n$ and are pairwise orthogonal with respect to the standard inner product on $\mathbb{C}^n$.*

A direct corollary of theorem is the following.

Corollary 2.1.1. *Suppose that A is both normal and r_A-row regular. Then there exists an orthonormal basis $\{v_1^A, v_2^A, \ldots, v_n^A\}$ of eigenvectors of A associated with the eigenvalues $\{\lambda_1^A, \lambda_2^A, \ldots, \lambda_n^A\}$ such that $v_1^A = \frac{1}{\sqrt{n}}\mathbb{1}_n = \frac{1}{\sqrt{n}}(1, \ldots, 1)^t \in \mathbb{C}^n$. In particular, $r_A = \lambda_1^A$ and, for $2 \leq k \leq n$, the standard inner product $\langle v_1^A, v_k^A \rangle = 0$.*

Another corollary is the following.

Corollary 2.1.2. *If A is both normal and r_A-row regular. Then A is also r_A-column regular. In particular, A is a semimagic square matrix.*

Proof. Let $\{v_1^A, v_2^A, \ldots, v_n^A\}$ be the system of orthonormal eigenvectors of A associated with the eigenvalues $\{r_A, \lambda_2^A, \ldots, \lambda_n^A\}$ as described in Corollary 2.1.1. Let $V = (v_1^A, v_2^A, \ldots, v_n^A)$ be the $n \times n$ matrix formed by this system of eigenvectors and let $D = \mathrm{diag}(r_A, \ldots, \lambda_n^A)$ be the digonal matrix of corresponding eigenvalues. We then have $AV = VD$. Since $\{v_1^A, v_2^A, \ldots, v_n^A\}$ is an orthonormal basis, we have $VV^* = V^*V = I_n$, and hence $V^* = V^{-1}$. Therefore, we can rewrite the equation $AV = VD$ as

$$(V^*)A = DV^*.$$

This shows that the rows of V^*, namely $\{(v_1^A)^*, (v_2^A)^*, \ldots, (v_n^A)^*\}$ form a system of orthnormal *left* eigenvectors for A associated with the eigenvalues $\{r_A, \lambda_2^A, \ldots, \lambda_n^A\}$. We conclude that the column sums of A are equal to $\lambda_1^A = r_A$ as well.

□

3. Joins of normal matrices with constant row sums

Let $d, k_1, k_2, \ldots, k_d \in \mathbb{N} \setminus \{0\}$, and set $n = k_1 + k_2 + \ldots + k_d$. Thus $\mathbf{k}_d = (k_1, \ldots, k_d)$ is a partition of n into d non-zero summands. Following [18], we shall consider $n \times n$ matrices of the following form

$$A = \left(\begin{array}{c|c|c|c} A_1 & a_{12}\mathbf{1} & \cdots & a_{1d}\mathbf{1} \\ \hline a_{21}\mathbf{1} & A_2 & \cdots & a_{2d}\mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d1}\mathbf{1} & a_{d2}\mathbf{1} & \cdots & A_d \end{array}\right), \qquad (*)$$

where, for each $1 \le i, j \le d$, A_i is a normal, r_{A_i}-row regular matrix of size $k_i \times k_i$ with complex entries, and $a_{i,j}\mathbf{1}$ is a $k_i \times k_j$ matrix with all entries equal to a constant $a_{i,j} \in \mathbb{C}$. These matrices will be called *$\mathbf{k}_d$-joins of normal row regular* (*NRR* for short) *matrices.*

For each $1 \le i \le d$, let $\{v_1^{A_i}, v_2^{A_i}, \ldots, v_{k_i}^{A_i}\}$ and $\{\lambda_1^{A_i}, \lambda_2^{A_i}, \ldots, \lambda_{k_i}^{A_i}\}$ be the set of eigenvectors and eigenvalues of A_i as described in Corollary 2.1.1. The next proposition is a direct generalization of [18, Proposition 10]. Before stating it, let us introduce the convenient notation

$$(x_1, \ldots, x_m)^T * (y_1, \ldots, y_n)^T = (x_1, \ldots, x_m, y_1, \ldots, y_n)^T.$$

For more vectors, we can define $*$ inductively.

Proposition 3.1. *For each $1 \le i \le d$ and $2 \le j \le k_i$ let*

$$w_{i,j} = \vec{0}_{k_1} * \ldots * \vec{0}_{k_{i-1}} * v_j^{A_i} * \vec{0}_{k_{i+1}} * \ldots * \vec{0}_{k_d}.$$

Then $w_{i,j}$ is an eigenvector of A associated with the eigenvalue $\lambda_j^{A_i}$.

Proof. By direct inspection, the key property being that, for $1 \le \ell \le d$, $\ell \ne i$ and $2 \le j \le k_i$, $\langle a_{\ell,i}\mathbb{1}_{k_i}, v_j^{A_i}\rangle = 0$, according to Corollary 2.1.1.

□

We will refer to the $w_{i,j}$'s and to the associated eigenvalues $\lambda_j^{A_i}$ as the old NRR eigenvectors and eigenvalues of A. Let $\lambda_1, \lambda_2, \ldots \lambda_d$ be the (not necessarily distinct) remaining eigenvalues of A.

Definition 3. The reduced characteristic polynomial of A is

$$\overline{p}_A(t) = \prod_{i=1}^{d}(t-\lambda_i) = \frac{p_A(t)}{\prod_{\substack{1\le i\le d,\\ 2\le j\le k_i}}(t-\lambda_j^{A_i})} = \frac{p_A(t)}{\prod_{i=1}^{d}\frac{p_{A_i}(t)}{t-r_{A_i}}}.$$

We will now describe $\overline{p}_A(t)$ as the characteristic polynomial of the matrix

$$\overline{A} = \begin{pmatrix} r_{A_1} & a_{12}k_2 & \cdots & a_{1d}k_d \\ a_{21}k_1 & r_{A_2} & \cdots & a_{2d}k_d \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}k_1 & a_{d2}k_2 & \cdots & r_{A_d} \end{pmatrix}.$$

For a vector $w = (x_1, \dots, x_d) \in \mathbb{C}^d$, we define

$$w^{\otimes} = (\underbrace{x_1, \dots, x_1}_{k_1 \text{ terms}}, \dots, \underbrace{x_d, \dots, x_d}_{k_d \text{ terms}})^t \in \mathbb{C}^n \tag{3.1}$$

Theorem 3.2. *The reduced characteristic polynomial of A coincides with the characteristic polynomial of $\overline{A}$, namely*

$$\overline{p}_A(t) = p_{\overline{A}}(t).$$

In other words

$$p_A(t) = p_{\overline{A}}(t) \prod_{\substack{1\le i\le d,\\ 2\le j\le k_i}}(t-\lambda_j^{A_i}).$$

Proof. Firstly, we note that, by construction, for any $v \in \mathbb{C}^d$ and any $\lambda \in \mathbb{C}$

$$\left[(\overline{A}-\lambda I)v\right]^{\otimes} = (A-\lambda I)v^{\otimes}. \tag{3.2}$$

Let λ be an eigenvalue of $\overline{A}$, and let $w = (x_1, \dots, x_d)$ be an associated generalized eigenvector, satisfying $(\overline{A}-\lambda I_d)^m w = 0$ for a suitable m. We will show, by induction on m, that $(A - \lambda I_n)^m w^{\otimes} = 0$. If $m = 1$, the assertion is a consequence of Equation (3.2). If $m > 1$, consider the vector $w' = (\overline{A}-\lambda I_d)w$, which satisfies $(\overline{A}-\lambda I_d)^{m-1}w' = 0$. By induction hypothesis, $(A-\lambda I_n)^{m-1}(w')^{\otimes} = 0$, therefore, thanks to Equation (3.2),

$$(A-\lambda I_n)^m w^{\otimes} = (A-\lambda I_n)^{m-1}\left((A-\lambda I_n)w^{\otimes}\right) = (A-\lambda I)^{m-1}(w')^{\otimes} = 0.$$

In other words, the generalized eigenspaces of $\overline{A}$ lift to (direct summands of) generalized eigenspaces of A. Now we observe that the NRR eigenvectors of A, together with the generalized eigenvectors of A of the shape $w^{\otimes}$, $w \in \mathbb{C}^d$, form a linearly independent set thanks to Corollary 2.1.1. Hence, by dimension counting, the eigenvalues of $\overline{A}$ are precisely the eigenvalues $\lambda_1, \dots, \lambda_d$ of A, with the correct multiplicity. Equivalently, $\overline{p}_A(t) = p_{\overline{A}}(t)$. □

Remark 3. After proving Theorem 3.2, we learned from ResearchGate that a special form of this theorem has been proved in [19, Theorem 2.1] and [2, Theorem 3]. We would like to take this chance to clarify the similarities and differences between our approaches. First, both our methods investigate the "broadcasting" mechanism to lift eigenvalues and eigenvectors from $\bar{A}$ to A as described by Eq. (3.1) (this broadcasting procedure has a physical interpretation as we explained in our work [16].) If A is symmetric, then A is diagonalizable and hence $\bar{A}$ is diagonalizable as well. In this case, [19, Theorem 2.1] only needs to deal with eigenvalues. Our method shows that the broadcasted solution described by Eq. (3.1) even works at the level of *generalized* eigenvalues. In other words, it even works for the cases where either $\overline{A}$ is not diagonalizable or A is not a symmetric matrix. This is important for applications because many networks in the Kuramoto models are directed.

Remark 4. We discuss a generalization of Theorem 3.2. More precisely, we can show that Theorem 3.2 holds for any field F under the mild assumption that k_i is invertible in F for all $1 \leq i \leq d$. In particular, we can drop the "normal" condition on A_i. First, we recall from Remark 2 that a $k_1 \times k_1$ matrix A_1 with entries in a field F is called a semimagic square if A_1 is both r_{A_1}-regular and c_{A_1}-regular and $c_{A_1} = r_{A_1}$. If k_1 is invertible in F, then F^{k_1} can be decomposed into

$$F^{k_1} = F\mathbb{1}_{k_1} \oplus W_1. \tag{3.3}$$

Here $F\mathbb{1}_{k_1}$ is the one dimensional vector space generated by $\mathbb{1}_{k_1}$ and W_1 is the set of all vectors $(x_1, x_2, \ldots, x_{k_1}) \in F^{k_1}$ such that $\sum_{i=1}^{k_1} x_i = 0$. We can check that each component of this decomposition is stable under A_1 for any semimagic square A_1. Now suppose that A is the join of d semimagic squares A_i of sizes $k_i \times k_i$ as defined in equation $*$. We assume that further that k_i is invertible in the field F. Let W_i be the decomposition

$$F^{k_i} = F\mathbb{1}_{k_i} \oplus W_i.$$

We see that for $1 \leq i \leq d$

$$\widehat{W_i} = \{\vec{0}_{k_1} * \ldots * \vec{0}_{k_{i-1}} * v_i * \vec{0}_{k_{i+1}} * \ldots * \vec{0}_{k_d} \quad | v_i \in W_i\},$$

is an A-stable subpsace of F^{k_i}. By the same proof as explained in Theorem 3.2, we can see that

$$\overline{p}_A(t) = p_{\overline{A}}(t). \tag{3.4}$$

We also note that the set of all such A with coefficients in any ring R has the structure of a ring (the case $d = 1$ was considered in [13]). By the same method described in the proof of [3, Theorem 3.16], we could describe the structure of this ring and derive Equation 3.4 as a direct consequence. We could show, in particular, that the map $A \mapsto \bar{A}$ is a ring homomorphism.

4. Applications to spectral graph theory

4.1. **Spectrum of the join of regular graphs.** In this section, we apply Theorem 3.2 to give new proofs for Theorem 2.1.8 and Theorem 2.1.9 in [6]. Let $G_1, G_2, \ldots, G_d$ be undirected regular graphs such that G_i has degree r_i and k_i vertices. Let G be the join graph of $G_1, G_2, \ldots, G_d$, which we will denote by $G = G_1 + G_2 + \ldots + G_d$. We recall that G is obtained from the disjoint union of $G_1, \ldots, G_2, \ldots, G_d$ by joining each vertex G_i with each vertex in all others G_j for $j \neq i$ (see [18, Section 4] and the reference therein for further details). Let A_i be the adjacency matrix of G_i for $1 \leq i \leq d$ and A be the adjacency matrix of G. By definition of the join of graphs, the adjacency matrix A of G has the following form

$$A = \left(\begin{array}{c|c|c|c} A_1 & \mathbf{1} & \cdots & \mathbf{1} \\ \hline \mathbf{1} & A_2 & \cdots & \mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{1} & \mathbf{1} & \cdots & A_d \end{array}\right).$$

Since G_i is an undirected graph, A_i is real and symmetric, hence normal. Furthermore, since G_i is regular of degree r_i, A_i is r_i-row regular. By Theorem 3.2, the reduced characteristics polynomial of A is given by

$$\overline{p}_A(t) = p_{\overline{A}}(t),$$

where

$$\overline{A} = \begin{pmatrix} r_1 & k_2 & \cdots & k_d \\ k_1 & r_2 & \cdots & k_d \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & r_d \end{pmatrix}.$$

In summary, we have

Proposition 4.1. *The characteristic polynomial of A is given by*

$$p_A(t) = p_{\overline{A}}(t)\frac{\prod_{i=1}^d p_{A_i}(t)}{\prod_{i=1}^d (t-r_i)}.$$

Let us consider some special cases of this proposition.

Corollary 4.1.1. *(See* [6, Theorem 2.1.8]*) If G_1 is r_1-regular with k_1 vertices and G_2 is r_2-regular with k_2 vertices then the characteristic polynomial of the join G_1+G_2 is given by*

$$p_{G_1+G_2}(t) = \frac{p_{G_1}(t)p_{G_2}(t)}{(t-r_1)(t-r_2)}\left((t-r_1)(t-r_2)-k_1k_2\right).$$

Proof. Let A_1, A_2 be the adjacency matrix of G_1, G_2 respectively. Then, the adjacency matrix of G_1+G_2 is

$$A = \begin{pmatrix} A_1 & \mathbf{1} \\ \mathbf{1} & A_2 \end{pmatrix}.$$

We have

$$\overline{A} = \begin{pmatrix} r_1 & k_2 \\ k_1 & r_2 \end{pmatrix}.$$

Hence

$$p_{\overline{A}}(t) = (t-r_1)(t-r_2) - k_1k_2.$$

By Proposition 3.1, we conclude that

$$p_{G_1+G_2}(t) = \frac{p_{G_1}(t)p_{G_2}(t)}{(t-r_1)(t-r_2)}\left((t-r_1)(t-r_2)-k_1k_2\right).$$

□

Corollary 4.1.2. *(See* [6, Theorem 2.1.9] *Let G_i be r_i-regular with k_i vertices. Assume further that*

$$k_1 - r_1 = k_2 - r_2 = \ldots = k_d - r_d = s.$$

Let G be the join graph of $G_1, G_2, \ldots, G_d$. Let

$$n = k_1 + k_2 + \ldots + k_d,$$

and

$$r = n - s.$$

Then

(1) *G is r-regular with n vertices.*
(2) *The characteristic polynomial of G is given by*

$$p_G(t) = (x-r)(x+s)^{d-1}\frac{\prod_{i=1}^d p_{G_i}(t)}{\prod_{i=1}^d (t-r_i)}.$$

Proof. Let v_i be a vertex in G_i. By definition, the degree of v_i in G is given by

$$\deg_{G_i}(v_i) + (n-k_i) = n - (k_i - r_i) = n - s = r.$$

We conclude that G is r-regular. This proves part (1). For part (2), we note that if A is the adjacency matrix of G then $\overline{A}$ is given by

$$\overline{A} = \begin{pmatrix} r_1 & k_2 & \cdots & k_d \\ k_1 & r_2 & \cdots & k_d \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & r_d \end{pmatrix}.$$

We observe that

$$\overline{A} + sI_d = \begin{pmatrix} k_1 & k_2 & \cdots & k_d \\ k_1 & k_2 & \cdots & k_d \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & k_d \end{pmatrix}$$

has rank 1. Consequently, $-s$ is an eigenvalue of $\overline{A}$ with multiplicity at least $d-1$. Additionally, by part (1), G is r-regular, hence $\lambda = r$ is the remaining eigenvalue of $\overline{A}$. Consequently,

$$p_{\overline{A}}(t) = (t-r)(t+s)^{d-1}.$$

By Proposition 4.1, we conclude that

$$p_G(t) = (t-r)(t+s)^{d-1} \frac{\prod_{i=1}^d p_{G_i}(t)}{\prod_{i=1}^d (t-r_i)}.$$

□

4.2. **A simple construction of Ramanujan graphs.** We discuss some applications of Corollary 4.1.2 to the construction of Ramanujan graphs. We first recall the definition of these graphs (see [6, Chapter 3] and [14] for further details.) We also recommend [9] for a beautiful survey of some surprising applications and occurrence of Ramanujan graphs in various parts of mathematics, physics, communications networks and computer science.)

Definition 4. (see [6, Definition 3.5.4]) Let G be a connected r-regular graph with k vertices, and let $r = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of G. Since G is connected and r-regular, its eigenvalues satisfy $|\lambda_i| \leq r, 1 \leq i \leq n$. Let

$$\lambda(G) = \max_{|\lambda_i|<r} |\lambda_i|.$$

The graph G is a *Ramanujan graph* if

$$\lambda(G) \leq 2\sqrt{r-1}.$$

The following proposition provides a construction of Ramanujan graphs.

Proposition 4.2. *Let $d \geq 2$ and, for $1 \leq i \leq d$, let G_i be r_i-regular Ramanujan graphs with k_i vertices. Suppose further that the G_i's satisfy the same conditions as in Corollary 4.1.2, namely*

$$k_1 - r_1 = k_2 - r_2 = \ldots = k_d - r_d = s.$$

Let G be the join graph of $G_1, G_2, \ldots, G_d$ and $n = k_1 + k_2 + \ldots + k_d$. Then G is a Ramanujan graph if and only if

$$s \leq 2(\sqrt{n} - 1).$$

Proof. Corollary 4.1.2 describes the eigenvalues of G. Taking into account that the valency r of G is greater than the valency r_i of each G_i, and that each G_i is Ramanujan, G is Ramanujan if and only if $s \leq 2\sqrt{r-1} = 2\sqrt{n-s-1}$, if and only if $s^2+4s-4n+4 \leq 0$, if and only if $s \leq 2\sqrt{n}-2$. □

Here is a special case of this construction.

Corollary 4.2.1. *Let G be a r-regular graph with k vertices. Let G^d be the join graph of d identical copies of G. Then there exists a natural number d_0 such that for all $d \geq d_0$, G^d is a Ramanujan graph.*

Proof. By Proposition 4.2, G^d is a Ramanujan graph if and only if

$$k - r \leq 2(\sqrt{dk} - 1).$$

This is equivalent to

$$d \geq \frac{1}{k}\left(\frac{k-r}{2} + 1\right)^2.$$

We therefore can take

$$d_0 = \left\lceil \frac{1}{k}\left(\frac{k-r}{2} + 1\right)^2 \right\rceil.$$

□

4.3. **Spectrum of the joined union of graphs.** Let G be a graph with d vertices $\{v_1, v_2, \ldots, v_d\}$. Let $G_1, G_2, \ldots, G_d$ be graphs. The joined union $G[G_1, G_2, \ldots, G_d]$ is obtained from the union of $G_1, \ldots, G_d$ by joining with an edge each pair of a vertex from G_i and a vertex from G_j whenever v_i and v_j are adjacent in G (see [19] for further details). Let $A_G = (a_{ij})$ be the adjacency matrix of G and $A_1, A_2, \ldots, A_d$ be the adjacency matrices of $G_1, G_2, \ldots, G_d$ respectively. The adjacency matrix of $G[G_1, G_2, \ldots, G_d]$ has the following form

$$A = \left(\begin{array}{c|c|c|c} A_1 & a_{12}\mathbf{1} & \cdots & a_{1d}\mathbf{1} \\ \hline a_{21}\mathbf{1} & A_2 & \cdots & a_{2d}\mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d1}\mathbf{1} & a_{d2}\mathbf{1} & \cdots & A_d \end{array}\right). \tag{4.1}$$

Remark 5. When $G = K_d$, the complete graph on d vertices, $G[G_1, G_2, \ldots, G_d]$ is exactly the join graph of $G_1, G_2, \ldots, G_d$ discussed in Section 4.1.

By Theorem 3.2, the spectrum of $G[G_1, G_2, \ldots, G_d]$ can be described by the spectra of G_i and an auxiliary matrix describing the interconnections between G_i. More precisely, we have the following proposition.

Proposition 4.3. *Assume that for each $1 \leq i \leq d$, G_i is a r_i-regular graph with k_i nodes. Let $G[G_1, G_2, \ldots, G_d]$ be the joined union graph. Let $\{\lambda_1^{G_i} = r_i, \ldots, \lambda_{k_i}^{G_i}\}$ be the spectrum of G_i as described in Corollary 2.1.1. Then the spectrum of A is the union of Spec($\overline{A}$) and the following multiset*

$$\{\lambda_j^{A_i}\}_{1 \leq i \leq d, 2 \leq j \leq k_i}.$$

Here $\overline{A}$ is the following $d \times d$ matrix, whose entries are the row sums of the blocks in the matrix A

$$\overline{A} = \begin{pmatrix} r_{A_1} & a_{12}k_2 & \cdots & a_{1n}k_d \\ a_{21}k_1 & r_{A_2} & \cdots & a_{2n}k_d \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}k_1 & a_{d2}k_2 & \cdots & r_{A_d} \end{pmatrix}.$$

Proof. This proposition follows from Theorem 3.2. To see this, we recall that the adjacency matrix of $G[G_1, G_2, \ldots, G_d]$ has the form described in Eq. (4.1) where A_G is the adjacency matrix of G and $A_1, A_2, \ldots, A_d$ are the adjacency matrices of $G_1, G_2, \ldots, G_d$ respectively. Since G_i is an undirected graph, we know that A_i is symmetric. Furthermore, by our assumption, G_i is r_i-regular, A_i is a normal, row regular with $r_i = r_{A_i}$. Therefore, we can apply Theorem 3.2 to obtain the above description for the spectrum of A. □

Let us consider another special case where the G_i are all r-regular graphs with k vertices. In this case, we have $k_1 = k_2 = \ldots = k_d = k$ and $r_{A_1} = r_{A_2} = \ldots = r_{A_d} = r$. Therefore, by Proposition 4.3, we have the following.

Proposition 4.4. *Assume that for each $1 \leq i \leq d$, G_i is a r-regular graph with k vertices. Let $G[G_1, G_2, \ldots, G_d]$ be the joined union graph. Let $\{\lambda_1^{G_i} = r, \ldots, \lambda_{k_i}^{G_i}\}$ be the spectrum of G_i as described in Corollary 2.1.1. Then the spectrum of A is the union the multiset*

$$\{\lambda_j^{A_i}\}_{1 \leq i \leq d, 2 \leq j \leq k_i},$$

and the following multiset

$$\{r + k\sigma | \sigma \in Spec(A_G)\}.$$

Proof. In this case, the matrix $\overline{A}$ is of the following form

$$\overline{A} = rI_d + kA_G,$$

where A_G is the adjacency matrix of G. Thus the spectrum of $\overline{A}$ consists of the roots of the characteristic polynomial

$$p_{\overline{A}}(t) = \det(tI_d - rI_d - kA_G).$$

Therefore, the spectrum of $\overline{A}$ is given by $r + k\,\mathrm{Spec}(A_G)$. □

4.4. **Energy of the joined union of graphs.** The concept of graph energy originates from problems in theoretical chemistry. Specifically, the mathematical definition of graph energy was inspired by early studies on the total π-electron energy of molecules represented by molecular graphs (see [5, 12, 7]). Interest in graph energy remained relatively dormant until around 2000, when a small group of mathematicians mutually found their interest in this topic, leading to an explosion of research. For a more detailed discussion on the historical development of graph energy, we refer the reader to the survey article [8].

Our interest in this topic arises from our experimental observation that the graph energy seems to increase when we apply the join operation on graphs. The goal of this section is to formalize this observation and propose a precise question about the relationship between the energy of the joined union of graphs and the energy of individual graphs (see Question 1).

We first recall the definition of energy of a graph.

Definition 5. Let G be a graph with d nodes. Suppose that

$$\mathrm{Spec}(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}.$$

The energy of G is defined to be the following sum (see [6, Section 9.2.2] for further discussions.)

$$E(G) = \sum_{i=1}^{d} |\lambda_i|.$$

***Example* 1.** If $G = K_d$ the complete graph with d vertices. Then

$$\mathrm{Spec}(G) = \{[-1]_{d-1}, [d-1]_1\},$$

where $[a]_m$ means that a has multiplicity m. We conclude that the energy of K_d is $2(d-1)$.

Let G_i and G be as at the beginning of Section 4.1, namely

$$G = G_1 + G_2 + \ldots + G_d = K_d[G_1, G_2, \ldots, G_d].$$

We have the following inequality.

Theorem 4.5. *The energy of G is strictly larger than the sum of the energy of G_i:*

$$E(G) > \sum_{i=1}^{d} E(G_i).$$

Proof. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ be the eigenvalues of $\overline{A}$ where A and $\overline{A}$ are the matrices defined at the beginning of Section 4.1, namely

$$\overline{A} = \begin{pmatrix} r_1 & k_2 & \cdots & k_d \\ k_1 & r_2 & \cdots & k_d \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & r_d \end{pmatrix}.$$

Note that $\lambda_i \in \mathbb{R}$ as they are also eigenvalues of A, which is real and symmetric. By Proposition 4.1, we have

$$E(G) - \sum_{i=1}^{d} E(G_i) = \sum_{i=1}^{d} |\lambda_d| - \sum_{i=1}^{d} r_i.$$

We also note that $\sum_{i=1}^{d} \lambda_i = \mathrm{Tr}(\overline{A}) = \sum_{i=1}^{d} r_i$. Therefore, we have

$$E(G) - \sum_{i=1}^{d} E(G_i) = \sum_{i=1}^{d} (|\lambda_i| - \lambda_i) = 2 \sum_{\lambda_i < 0} |\lambda_i|.$$

Hence, to show that $E(G) > \sum_{i=1}^{d} E(G_i)$, we only need to show that for some i, $\lambda_i < 0$.

Let $s_i = k_i - r_i > 0$. Without loss of generality, we can assume that

$$k_1 - r_1 \leq k_2 - r_2 \leq \ldots \leq k_d - r_d.$$

Let us consider

$$\begin{aligned} p_{\overline{A}}(-s_1) = p_{\overline{A}}(r_1 - k_1) &= \det((r_1 - k_1) - \overline{A}) \\ &= (-1)^d \det \begin{pmatrix} k_1 & k_2 & \cdots & k_d \\ k_1 & r_2 + k_1 - r_1 & \cdots & k_d \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & r_d + k_1 - r_1 \end{pmatrix} \\ &= (-1)^d k_1 \det \begin{pmatrix} 1 & k_2 & \cdots & k_d \\ 1 & r_2 + k_1 - r_1 & \cdots & k_d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & k_2 & \cdots & r_d + k_1 - r_1 \end{pmatrix}. \end{aligned}$$

By adding $-k_i$ times the first column to the i-th column, we see that the later determinant is also equal to

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & (k_1 - r_1) - (k_2 - r_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & (k_1 - r_1) - (k_d - r_d) \end{pmatrix} = (s_1 - s_2)(s_1 - s_3) \ldots (s_1 - s_d).$$

We conclude that

$$p_{\overline{A}}(-s_1) = (-1)^d k_1 \prod_{j \neq 1} (s_1 - s_j) = -k_1 \prod_{j \neq 1} (s_j - s_1) \leq 0.$$

By the same argument, we see that

$$p_{\overline{A}}(-s_2) = -k_2 \prod_{j\neq 2}(s_j - s_2) = k_2(s_2 - s_1)\prod_{j>2}(s_j - s_2) \geq 0.$$

By the mean value theorem, $p_{\overline{A}}(t)$ has a real root on the interval $[-s_2, -s_1]$. In particular, at least one eigenvalue of $\overline{A}$ must be negative. This completes the proof. □

Definition 6. A graph G with d nodes is called hyperenergetic if $E(G) \geq 2(d-1)$.

Theorem 4.6. *Assume that G_i are all r-regular with k vertices. Assume further that G is hyperenergetic. Then*

$$E(G[G_1, G_2, \dots, G_d]) \geq E(G) + \sum_{i=1}^{d} E(G_i).$$

The equality can happen, for example when G and G_i are all complete graphs.

Proof. Let A be the adjacency matrix of $G[G_1, G_2, \dots, G_d]$. Then the matrix $\overline{A}$ in Proposition 4.3 has the following form

$$\overline{A} = \begin{pmatrix} r & a_{12}k & \cdots & a_{1n}k \\ a_{21}k & r & \cdots & a_{2n}k \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}k & a_{d2}k & \cdots & r \end{pmatrix} = rI_d + kA_G.$$

Let $\mathrm{Spec}(A_G) = \{\lambda_1, \lambda_2, \dots, \lambda_d\}$ then

$$\mathrm{Spec}(\overline{A}) = \{r + k\lambda_1, r + k\lambda_2, \dots, r + k\lambda_d\}.$$

By Proposition 4.3, we have

$$E(G[G_1, G_2, \dots, G_d]) - E(G) - \sum_{i=1}^{d} E(G_i) = \sum_{i=1}^{d} |r + k\lambda_i| - \sum_{i=1}^{d} |\lambda_i| - dr.$$

We note that by the Perron-Frobenius Theorem, one of the eigenvalues of A_G must be real and non-negative. Let us assume $\lambda_1 \geq 0$. We then have

$$\begin{aligned} \sum_{i=1}^{d} |r + k\lambda_i| &= r + k\lambda_1 + \sum_{i=2}^{d} |r + k\lambda_i| \\ &\geq r + k\lambda_1 + \sum_{i=2}^{d} (k|\lambda_i| - r) \\ &\geq k\sum_{i=1}^{d} |\lambda_i| - (d-2)r. \end{aligned}$$

Consequently, we have

$$\begin{aligned} E(G[G_1, G_2, \dots, G_d]) - E(G) - \sum_{i=1}^{d} E(G_i) &\geq (k-1)\sum_{i=1}^{d} |\lambda_i| - 2(d-1)r \\ &\geq r(\sum_{i=1}^{d} |\lambda_i| - 2(d-1)) \\ &\geq 0. \end{aligned}$$

Note that the second inequality follows from $k \geq r+1$ and the last inequality follows from the assumption that G is hyperenergetic. □

Remark 6. The above proof can be slightly generalized as follows. Suppose that G is an undirected graph and the spectrum of G consists of n negative eigenvalues and p non-negative eigenvalues. Suppose that the energy of G satisfies

$$E(G) \geq d + n - p = 2(d-p). \tag{4.2}$$

Then we have

$$E(G[G_1, G_2, \ldots, G_d]) \geq E(G) + \sum_{i=1}^{d} E(G_i).$$

We checked that all undirected graphs with at most 3 nodes satisfy the Inequality 4.2.

***Question* 1.** Suppose that G_i are all regular graphs. Does the following inequality hold in general?

$$E(G[G_1, G_2, \ldots, G_d]) \geq E(G) + \sum_{i=1}^{d} E(G_i)? \tag{4.3}$$

We provide an answer to this question in a special case, namely for $d = 2$.

Proposition 4.7. *Let G_1, G_2 be two regular graphs and G be a graph with 2 nodes. Then*

$$E(G[G_1, G_2]) \geq E(G) + E(G_1) + E(G_2).$$

Proof. If G is the cocomplete graph, we have

$$E(G[G_1, G_2]) = E(G) + E(G_1) + E(G_2).$$

Suppose now that $G = K_2$ the complete graph on 2 nodes. The energy of G is $E(G) = 2$. Suppose that G_i is r_i regular with k_i vertices for $i \in \{1, 2\}$. Let λ_1, λ_2 be the eigenvalues of $\overline{A}$ where

$$\overline{A} = \begin{pmatrix} r_1 & k_2 \\ k_1 & r_2 \end{pmatrix}.$$

By Proposition 4.1 we have

$$E(G[G_1, G_2]) - E(G_1) - E(G_2) = |\lambda_1| + |\lambda_2| - (r_1 + r_2).$$

We conclude that

$$\lambda_1, \lambda_2 = \frac{(r_1 + r_2) \pm \sqrt{(r_1 - r_2)^2 + 4k_1k_2}}{2}.$$

We have $\det(\overline{A}) = r_1 r_2 - k_1 k_2 < 0$ so one root of $\overline{A}$ is negative and the other is positive. Consequently

$$\begin{aligned} |\lambda_1| + |\lambda_2| - r_1 - r_2 &= \sqrt{(r_1 - r_2)^2 + 4k_1k_2} - (r_1 + r_2) \\ &\geq \sqrt{(r_1 - r_2)^2 + 4(r_1 + 1)(r_2 + 1)} - (r_1 + r_2) \\ &\geq (r_1 + r_2 + 2) - (r_1 + r_2) = 2. \end{aligned}$$

In other words, we have

$$E(G[G_1, G_2]) \geq E(G) + E(G_1) + E(G_2).$$

□

Another situation where we can verify Inequality 4.3 is the following.

Proposition 4.8. *Let G_i be r_i-regular with k_i vertices. Assume further that*

$$k_1 - r_1 = k_2 - r_2 = \ldots = k_d - r_d = s.$$

Let G be the joined union graph $K_d[G_1, G_2, \ldots, G_d]$. Then

$$E(K_d[G_1, G_2, \ldots, G_d]) \geq E(K_d) + \sum_{i=1}^{d} E(G_i).$$

Proof. Let $k = \sum_{i=1}^{d} k_i$. By Corollary 4.1.2, we have

$$\begin{aligned}
&E(K_d[G_1, G_2, \ldots, G_d]) - E(K_d) - \sum_{i=1}^{d} E(G_i) \\
&= (k - s) + (d-1)s - 2(d-1) - \sum_{i=1}^{d} r_i \\
&= \sum_{i=1}^{d} (k_i - r_i) - s + (d-1)(s-2) \\
&= ds - s + (d-1)(s-2) = 2(d-1)(s-1) \geq 0.
\end{aligned}$$

Consequently

$$E(K_d[G_1, G_2, \ldots, G_d]) \geq E(K_d) + \sum_{i=1}^{d} E(G_i).$$

□

Proposition 4.9. *Let G_i be r_i-regular with k_i vertices. Let $s_i = k_i - r_i$ Assume further that*

$$s_1 < s_2 < \ldots < s_d.$$

Let G be the joined union graph $K_d[G_1, G_2, \ldots, G_d]$. Then

$$E(K_d[G_1, G_2, \ldots, G_d]) \geq 2\sum_{i=1}^{d-1} s_i + \sum_{i=1}^{d} E(G_i).$$

In particular, if $d \geq 2$ then

$$E(K_d[G_1, G_2, \ldots, G_d]) > E(K_d) + \sum_{i=1}^{d} E(G_i).$$

Proof. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ be the eigenvalues of $\overline{A}$ where A and $\overline{A}$ are the matrices in Proposition 4.1, namely

$$\overline{A} = \begin{pmatrix} r_1 & k_2 & \cdots & k_d \\ k_1 & r_2 & \cdots & k_d \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & r_d \end{pmatrix}.$$

By the same argument as in Proposition 4.5, we have

$$p_{\overline{A}}(-s_i) = -k_i \prod_{j \neq i} (s_j - s_i).$$

Because of the total ordering $s_1 < s_2 < \ldots < s_d$, we see that $p_{\overline{A}}(-s_i)p_{\overline{A}}(-s_{i+1}) < 0$ for $1 \leq i \leq d-1$. By the mean value theorem, $p_{\overline{A}}(t)$ has a real root, say λ_i, in the interval $[-s_{i+1}, -s_i]$. In particular, $\lambda_i < 0$ and $|\lambda_i| \geq s_i$ for $1 \leq i \leq d-1$. We also note that

$$\sum_{i=1}^{d} \lambda_i = \mathrm{Tr}(\overline{A}) = \sum_{i=1}^{d} r_i.$$

Hence

$$\lambda_d = \sum_{i=1}^{d} r_i - \sum_{i=1}^{d-1} \lambda_i > 0.$$

We then have

$$\begin{aligned} E(K_d[G_1, G_2, \ldots, G_d]) - \sum_{i=1}^{d} E(G_i) &= \sum_{i=1}^{d} |\lambda_i| - \sum_{i=1}^{d} r_i \\ &= \sum_{i=1}^{d-1} |\lambda_i| + \left(\sum_{i=1}^{d} r_i - \sum_{i=1}^{d-1} \lambda_i \right) - \sum_{i=1}^{d} r_i \\ &= 2\sum_{i=1}^{d-1} |\lambda_i| \geq 2\sum_{i=1}^{d-1} s_i. \end{aligned}$$

Since $1 \leq s_1 < s_2 < \ldots < s_d$, the above inequality implies that

$$E(K_d[G_1, G_2, \ldots, G_d]) - \sum_{i=1}^{d} E(G_i) > 2(d-1) = E(K_d).$$

□

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