SUPERCHARACTER THEORY AND APPLICATIONS TO RAMANUJAN SUMS OVER A FINITE FROBENIUS RING

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ABSTRACT. The theory of Ramanujan sums has been playing a fundamental role in several subfields of mathematics. They appear in the theory of special values of zeta functions, spectral graph theory, representation theory, and analytic number theory. Recent work has shown that classical Ramanujan sums can also be interpreted as a super-Fourier transform via the theory of supercharacters for the ring $\mathbb{Z}/n$. In this article, building upon our recent work on supercharacters over an arbitrary finite Frobenius ring, we explore additional arithmetical properties of Ramanujan sums. Our approach provides a unified framework that generalizes various results in the literature regarding these sums. As a by-product, we also describe a new criterion for determining when a finite commutative ring is Frobenius, which could be of independent interest to the algebraist community.

1. INTRODUCTION

Let n be a positive integers and ζ_n be a fixed primitive n-root of unity. The sum

$$c_n(m) = \sum_{\substack{1\leq j\leq n \\ \gcd(j,n)=1}} \zeta_n^{mj}, \tag{1.1}$$

is known in the literature as a Ramanujan sum. Although Dirichlet and Dedekind had considered this sum in the mid-19th century, it was Ramanujan who first realized its importance in the early 20th century, using it to investigate several problems in number theory ([8, Page 159]). For instance, Ramanujan used these sums to derive new expressions for arithmetical functions such as the divisor function. Ramanujan sums have since found applications in various subfields of mathematics, including representation theory, analytical number theory, sieve theory, graph theory, and physics. In graph theory, for example, these sums have been used to study the spectra of certain classes of graphs. We refer the reader to [7, Section 1.1] for a more extensive discussion on the history and applications of Ramanujan sums.

Our own interest in Ramanujan sums stems from their recurring appearance in our research. These sums, together with Gauss sums, first appear in our calculations of

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the spectrum of the generalized Paley graph associated with a quadratic character (see [13]). They reappear again in our investigation of Fekete polynomials associated with principal Dirichlet characters. More precisely, we recall that the n-th Fekete polynomial is defined as

$$F_n(x) = \sum_{1 \leq a \leq n \ \gcd(a,n)=1} x^a. \tag{1.2}$$

By definition, $c_n(m)$ is precisely the value of F_n at an n-root of unity; namely $F_n(\zeta_n^m)$. Using the explicit formula for $c_n(m)$, we can show that, if n is squarefree and d is a divisor of n, the cyclotomic polynomial Φ_d is not a factor of F_n (see [4, Corollary 2.7]). Later on, while working on prime Cayley graphs (see [5]), we found the work of Klotz-Sander and So on unitary graphs and gcd-graphs where Ramanujan sums play central roles (see [11, 22]). For example, in [22], So uses Ramanujan sums to classify all integral circulant graphs, which effectively resolves a conjecture of Klotz and Sander on integral circulant graphs stated in [11]. While reading more work on gcd-graphs, we soon realized that the theory of gcd-graphs can be generalized to an arbitrary finite commutative ring. Furthermore, when the underlying ring is a Frobenius ring, we can even develop a general theory of Ramanujan sums and utilize them to calculate explicitly the spectra of the associated gcd-graphs This circle of ideas has led us to various work in this research direction (see [15, 17, 16, 19]). In this article, building upon recent advances on supercharacter theory and its applications to classical Ramanujan sums (see [6, 7, 20]), we study some further arithmetical properties of generalized Ramanujan sums. Along the way, we discuss some connections with spectral graph theory. Additionally, we investigate the determinant of a matrix associated with Ramanujan sums. Using the theory of supercharacters for Frobenius rings developed in [20], we determine the precise value of this determinant.

We remark that the generalized theory of Ramanujan sums over Frobenius rings was pioneered by Lamprecht in his 1953 work [12]. However, his contributions in this area have not received widespread recognition as it should in the mathematical community (see [9] for some further valuable historical context on Lamprecht's work on Frobenius rings). Given the significance of Lamprecht's early insights, we feel it is important to acknowledge his foundational role in this line of research.

1.1. **Outline.** The article is structured as follows. In Section 2.1, we review the theory of supercharacters over a finite abelian group. The key ideas in this section have been previously discussed in [7]. Our main contribution here is the introduction of certain related sums, which appear naturally in the spectral description of certain Cayley graphs. In Section 2.2, we apply the results from the previous section to the case where R is a finite Frobenius ring. More precisely, we explain the existence of a natural supercharacter theory on R and show how this theory is closely related to the theory of Ramanujan sums developed in [17]. Using the general results in Section 2.1, we derive various orthogonality relations for these Ramanujan sums. We also explain how our results recover some well-known formulas in the literature. Furthermore, we give an explicit formula for the k-moment of these Ramanujan sums for each $k \geq 1$. In Section 3, we explicitly describe the supercharacter table for the associated supercharacter theory outlined in Section 2.2. Additionally, we show that determinant of this supercharacter table can determine whether a finite commutative ring is Frobenius or not. Our main theorem provides a unified proof for various special cases studied in [21] (for the case where R is $\mathbb{Z}/n$) and [14] (for the case where R is a quotient of $\mathbb{F}_q[x]$). Finally, in Section 4, we provide a generalization of Kluyver's formula, which also unifies various scattered results in the literature.

2. Frobenius rings and their supercharacter theories

2.1. **Supercharacter theory and supercharacter table of a finite abelian group.** We first recall the definition of a supercharacter theory on a finite abelian group G. We refer the reader to [2, 6, 7] for further discussions on this topic. We remark that, in order to keep a consistent set of notations, our discussion here closely aligns with [7, Section 2].

Definition 2.1. A supercharacter theory on G is a pair $(\mathcal{K}, \mathcal{X})$ where $\mathcal{K} = \{K_1, K_2, \ldots, K_m\}$ be a partition of G and $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ a partition of the dual group $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^{\times})$ of characters of G which satisfies the following conditions

(1) $\{0\} \in \mathcal{K}$;
(2) $|\mathcal{X}| = |\mathcal{K}|$;
(3) For each $X_i \in \mathcal{X}$, the character sum
$$\sigma_i = \sum_{\chi \in X_i} \chi$$
is constant on each $K \in \mathcal{K}$;
(4) As explained in [20, Section 2], to do spectral theory for certain associated graphs, it is beneficial to add one more condition to $(\mathcal{K}, \mathcal{X})$; namely for a fixed $X \in \mathcal{X}$ the sum $\sum_{k \in K_i} \chi(k)$ does not depend on the choice of $\chi \in X$. We will denote this sum by $\Omega_{K_i}(X)$ or more simply $\Omega_i(X)$.

For a supercharacter theory $(\mathcal{K}, \mathcal{X})$, the corresponding supercharacter table is the $m \times m$ matrix $S = (\sigma_i(K_j))_{i,j=1}^{m}$. More precisely,

$$S = \begin{array}{c|cccc} & K_1 & K_2 & \cdots & K_m \\ \hline \sigma_1 & \sigma_1(K_1) & \sigma_1(K_2) & \cdots & \sigma_1(K_m) \\ \sigma_2 & \sigma_2(K_1) & \sigma_2(K_2) & \cdots & \sigma_2(K_m) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_m & \sigma_m(K_1) & \sigma_m(K_2) & \cdots & \sigma_m(K_m) \end{array} \tag{2.1}$$

As explained in [7], the matrix S satisfies several orthogonality properties. To describe these properties, we recall some concepts. First, the space of complex-valued functions $f\colon G \to \mathbb{C}$ is equipped with the following natural inner product.

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}. \tag{2.2}$$

A function $f\colon G \to \mathbb{C}$ is called a superclass function if f is constant on each superclass in $\{K_1, K_2, \dots, K_m\}$. For a superclass function f, we will denote by $f(K_i)$ the value of f at an element $x \in K_i$. The space of all superclass functions with respect to the pair $(\mathcal{K}, \mathcal{X})$ will be denoted by $\mathcal{S}$. This space $\mathcal{S}$ inherits the inner product structure from Eq. (2.2). For any two functions $f_1, f_2 \in \mathcal{S}$, their inner product can be expressed more concisely as:

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{\ell=1}^{m} |K_\ell| f_1(K_\ell)\overline{f_2(K_\ell)}. \tag{2.3}$$

As explained in [7], $\{\sigma_i\}_{i=1}^m$ forms an orthogonal basis for $\mathcal{S}$. More precisely, we have

$$\langle \sigma_i, \sigma_j\rangle = |X_i|\delta_{i,j}, \tag{2.4}$$

where δ is the Kronecker function and $|X_i|$ is the size of X_i since

$$|X_i| = ||X_i||_2^2 := \sqrt{\sum_{\chi\in X_i} |\chi(1)|^2}.$$

By Eq. (2.3) and Eq. (2.4) we have the following formula (see [7, Equation 2.4])

$$\frac{1}{|G|}\sum_{\ell=1}^{m} |K_\ell|\sigma_i(K_\ell)\overline{\sigma_j(K_\ell)} = |X_i|\delta_{i,j}. \tag{2.5}$$

Let

$$D = \operatorname{diag}\left(\sqrt{|K_1|}, \dots, \sqrt{|K_m|}\right),$$

$$L = \frac{1}{\sqrt{|G|}}\operatorname{diag}\left(\frac{1}{\sqrt{|X_1|}}, \dots, \frac{1}{\sqrt{|X_m|}}\right).$$

Then by Eq. (2.5), we have

$$(SD)(SD)^* = |G|\operatorname{diag}\left(|X_1|, \dots, |X_m|\right).$$

Furthermore, if we let $U = LSD$ then

$$U = \frac{1}{\sqrt{|G|}}\left[\frac{\sigma_i(K_j)\sqrt{|K_j|}}{\sqrt{|X_i|}}\right]_{i,j=1}^{n}$$

and $UU^* = I$; namely U is a unitary matrix. Since $U^*U = I$, we obtain the following orthogonality condition.

$$\frac{\sqrt{|K_i||K_j|}}{|G|} \sum_{\ell=1}^{m} \frac{\sigma_\ell(K_i)\overline{\sigma_\ell(K_j)}}{|X_\ell|} = \delta_{i,j}. \tag{2.6}$$

We state here a simple corollary of the fact that U is unitary, which we will use later on.

Proposition 2.2. *We have the following equality*

$$\det(S)^2 = |G|^m \prod_{i=1}^{\ell} \frac{|X_i|}{|K_i|}.$$

In particular, if $|K_i| = |X_i|$ *for each* $1 \leq i \leq m$*, then* $|\det(S)| = G^{\frac{m}{2}}$.

We remark that by [20, Proposition 2.3], we have

$$\frac{\Omega_j(X_i)|}{|K_j|} = \frac{\sigma_i(K_j)}{|X_i|}.$$

As a result, each orthogonality relation for $\sigma_i(K_j)$ can be converted to an equivalent one for $\Omega_j(K_i)$ and vice versa. For example, we can rewrite U as

$$U = \frac{1}{\sqrt{|G|}} \left[\frac{\Omega_j(X_i)\sqrt{|X_i|}}{\sqrt{|K_j|}} \right]_{i,j=1}^{n},$$

and Eq. (2.6) is equivalent to

$$\frac{1}{|G|\sqrt{|K_i||K_j|}} \sum_{\ell=1}^{m} |X_\ell|\Omega_i(X_\ell)\overline{\Omega_j(X_\ell)} = \delta_{i,j}. \tag{2.7}$$

We remark that since we assume $K_i = -K_i$, $\Omega_j(X_\ell) \in \mathbb{R}$ (see [20, Proposition 2.3]). As a result, Eq. (2.7) is equivalent to

$$\frac{1}{|G|\sqrt{|K_i||K_j|}} \sum_{\ell=1}^{m} |X_\ell|\Omega_i(X_\ell)\Omega_j(X_\ell) = \delta_{i,j}. \tag{2.8}$$

In particular, when $i = j$, we have

$$\frac{1}{|K_i||G|} \sum_{\ell=1}^{m} |X_\ell|\Omega_i(X_\ell)^2 = 1. \tag{2.9}$$

2.2. **Orthogonality relations for Ramanujan sums over a finite Frobenius ring.** In this section, we apply the results to the case where the abelian group G is the additive structure of a finite ring R. Here, we exploit the fact that a ring has another structure; namely the multiplicative structure. In general, it is unclear how to construct a supercharacter theory for a finite commutative ring. However, as explained [20], there is a class of rings that such a theory naturally exists; namely the class of finite commutative Frobenius ring. We first recall this concept.

Definition 2.3. Let R be a finite commutative ring. We say that R is a Frobenius ring if R is a $\mathbb{Z}/n$-algebra equipped with a non-degenerate $\mathbb{Z}/n$ linear functional $\psi : R \to \mathbb{Z}/n$. Here, non-degenerate means that the kernel of ψ does not contain any non-zero ideal in R.

By [9, 12], there are some other equivalent characterizations of a finite Frobenius ring. For example, a local ring is Frobenus iff its socle module is cyclic. In other words, there exists an element $e \in R \setminus \{0\}$ such that Re is contained in all non-zero ideals in R. A finite commutative ring is Frobenius if and only if it is a product of finite local Frobenius ring.

Let $\zeta_n := e^{\frac{2\pi i}{n}}$ be a fixed primitive n-root of unity and $\chi : R \to \mathbb{C}^\times$ be the character defined by $\chi(a) = \zeta_n^{\psi(a)}$. By [16, Proposition 2.4], the dual group $\mathrm{Hom}(R, \mathbb{C}^\times)$ is a cyclic R-module generated by χ; namely every character of R is of the form χ_r where $\chi_r(a) = \chi(ra)$. We note that by the definition of χ, the following identity holds for all $x, y \in R$.

$$\chi_x(y) = \chi_y(x) = \chi_{xy}(1) = \chi(xy).$$

We now recall the definition of the Ramanujan sum $c(g, R)$

Definition 2.4. Let $g \in R$. The generalized Ramanujan sum $c(g, R)$ is defined as follows

$$c(g, R) = c_\psi(g, R) = \sum_{a \in R^\times} \chi_g(a) = \sum_{a \in R^\times} \chi(ga).$$

We have two remarks about this definition. First, $c_n(m)$ is precisely $c(m, \mathbb{Z}/n)$. Therefore, $c(g, R)$ is a natural generalization of the classical Ramanujan sum defined in Eq. (1.1). Second, at first glance, it seems that $c_\psi(g, R)$ depends on ψ. However, as explained in [18, Theorem 4.14], $c_\psi(g, R)$ is independent of ψ. In fact, we have

$$c(g, R) = \frac{\varphi(R)}{\varphi(R/\mathrm{Ann}_R(g))} c(1, R/\mathrm{Ann}_R(g)) = \frac{\varphi(R)}{\varphi(R/\mathrm{Ann}_R(g))} \mu(R/\mathrm{Ann}_R(g)). \tag{2.10}$$

Here $\mathrm{Ann}_R(g)$ is the annihilator ideal of g; namely

$$\mathrm{Ann}_R(g) = \{r \in R \mid gr = 0\}.$$

Additionally, φ is the generalized Euler function; $\varphi(R) = |R^\times|$. Finally, μ is the generalized Möbius function defined as follows. We recall that by the Artin-Wedderburn structure theorem for Artinian rings, R is isomorphic to a finite product of local rings $R \cong \prod_{i=1}^d R_i$. Then, $\mu(R)$ is defined by the following rule.

$$\mu(T) = \begin{cases} 1, & \text{if } |T| = 1, \\ 0, & \text{if there exists } 1 \le i \le d \text{ such that } R_i \text{ is not a field,} \\ (-1)^d, & \text{otherwise.} \end{cases}$$

Let $K_1, K_2, \ldots, K_m$ be the orbits of R under the action of $R^\times$. We will define $\tau(R) = m$ since in the case $R = \mathbb{Z}/n$, $\tau(R) = \tau(n)$–the number of positive divisors of n. By definition, the elements in each K_i are associates. Without loss of generality, we will assume that $K_1 = R^\times$. For each $1 \le i \le m$, let

$$X_i = \{\chi_g | g \in K_i\}.$$

For convenience, we will denote by K_g (respectively X_g) the class that contains g (respectively χ_g). This is consistent with our convention that $K_1 = R^\times$. Additionally, we will use the notations $\sigma_x(K_y)$ and $\Omega_x(K_y)$ for the appropriate sums. With these preparations, we are now able to recall the following proposition.

Proposition 2.5. *(See* [20, Theorem 4.1]*) The pair* $(\mathcal{K}, \mathcal{X})$ *where* $\mathcal{K} = \{K_1, K_2, \ldots, K_m\}$ *and* $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ *is a supercharacter theory for* $(R, +)$*. Furthermore, for each* $1 \le i \le m$

$$|X_i| = |K_i| = \varphi(R/\mathrm{Ann}_R(g_i)),$$

where g_i *is an element in* K_i*.*

Proof. The fact that $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory is a direct consequence of [20, Theorem 4.1]). Let us now prove the second part about the size of $|K_g| = |X_g|$. Let $\mathrm{Stab}(g)$ be the stabilizer of g. We have

$$\mathrm{Stab}(x) = \{u \in R^\times | ux = u\} = \{u \in U | (u-1) \in \mathrm{Ann}_R(x)\} = \ker(R^\times \to (R/\mathrm{Ann}_R(g))^\times).$$

By the stabilizer theorem, we have $|K_g| = \dfrac{\varphi(R)}{\text{Stab(x)}}$ and hence by the first isomorphism of groups

$$|K_g| = |V_g| = \varphi(R/\mathrm{Ann}_R(g)).$$

□

We now explain the connection between the Ramanujan sums $c(g, R)$ and the values $\sigma_i(K_j)$ and $\Omega_i(K_j)$ described in Section 2.1. By definition,

$$c(g, R) = \Omega_{K_1}(X_g) = \Omega_1(X_g).$$

We now use the orthogonality properties described in Section 2.1 to derive several arithmetical properties of Ramanujan sums. We first have the following two theorems, which generalize a result of Carmichael in [3] for classical Ramanujan sums.

Theorem 2.6.
$$\sum_{g\in R} c(g,R)^2 = |R|\varphi(R),$$
where $\varphi(R) = |K_1| = |R^\times|$.

Proof. Using the facts that $|X_i| = |K_i|$, $c(g,R) = \Omega_1(X_g)$ and Eq. (2.9), we have

$$\sum_{g\in R} c(g,R)^2 = \sum_{\ell=1}^{m}\left[\sum_{g\in K_\ell} c(g,R)^2\right]$$
$$= \sum_{\ell=1}^{m} |K_\ell|\Omega_1(X_\ell)^2 = \sum_{\ell=1}^{m} |X_\ell|\Omega_1(X_\ell)^2 = |K_1||R| = \varphi(R)|R|.$$

□

While the statement the first orthogonality condition described in Theorem 2.6 follows directly from supercharacter theory, we can also prove it using a direct and elementary argument.

Proof. We have

$$\sum_{g\in R} c(g,R)^2 = \sum_{g\in R}\left(\sum_{a\in R^\times}\chi(ga)\right)\left(\sum_{b\in R^\times}\chi(gb)\right)$$
$$= \sum_{a\in R^\times}\sum_{b\in R^\times}\sum_{g\in R}\chi(g(a+b)) = \sum_{a\in R^\times}\sum_{b\in R^\times}\sum_{g\in R}\chi_{a+b}(g)$$

The inner sum is 0 when $a+b \neq 0$ and it is $|R|$ when $a+b=0$. Once a is fixed, there is a unique b such that $a+b=0$, therefore

$$\sum_{g\in R} c(g,R)^2 = \varphi(R)|R|.$$

□

We describe another proof for Theorem 2.6 using graph theory.

Proof. We recall that the unitary Cayley graph $G_R = \Gamma(R, R^\times)$ is the graph with the following data

(1) The vertex set of G_R is R.
(2) Two vertices a, b are adjacent if and only if $a - b \in R^\times$.

As explained in [17, Theorem 4.6] (see also [20, Theorem 4.6] for a generalization) the spectrum of G_R is precisely $\{c(g,R)\}_{g\in R}$. By the walk-counting formula, we have

$$\sum_{g\in R} c(g,R)^2 = \sum_{g\in R}\deg_{G_R}(g) = \varphi(R)|R|.$$

□

We discuss a slight generalization of Theorem 2.6. First, we introduce the following definition.

Definition 2.7. We define k-th moment of Ramanujan sums as

$$M_k(R) = \sum_{g \in G} c(g, R)^k.$$

By definition, we have $M_0(R) = |R|, M_1(R) = 0, M_2(R) = \varphi(R)|R|$.

Remark 2.8. If R_1 and R_2 are two Frobenius rings, then $M_k(R_1) = M_k(R_2)$ for all $k \geq 0$ if and only if G_{R_1} and G_{R_2} are cospectral. In particular, this happens if G_{R_1} and G_{R_2} are isomorphic. [10, Theorem 5.3] and [14, Theorem 4.1] show that these two graphs are isomorphic if and only $|R_1| = |R_2|$ and $R_1^{\mathrm{ss}} \cong R_2^{\mathrm{ss}}$. Here $R^{\mathrm{ss}} = R/\mathrm{Rad}(R)$ with $\mathrm{Rad}(R)$ being the Jacobson radical of R. In [14, Proposition 4.5], we show some examples of R_1, R_2 which are not isomorphic but their associated unitary Cayley graphs are. In fact, when these rings are a finite quotient of $\mathbb{F}_q[x]$, [14, Proposition 4.5] also determines the number of isomorphism classes of unitary Cayley graphs.

Proposition 2.9. *Let $R = \prod_{i=1}^d R_i$ is a product of finite local Frobenius rings. For each $1 \leq i \leq d$, let f_i be the order of the residue field of R_i. Then*

$$M_k(R) = |R|^k \prod_{i=1}^d \left[\left(1 - \frac{1}{f_i}\right)^k + (-1)^k \frac{f_i - 1}{f_i^k} \right]$$

Proof. We observe that $M_k(R)$ is multiplicative with respect to direct products; namely $M_k(R_1)M_k(R_2) = M_k(R_1 \times R_2)$. This follows directly from the property that for $r_1 \in R_1$ and $r_2 \in R_2$

$$c((r_1, r_2), R_1 \times R_2) = c(r_1, R_1)c(r_2, R_2).$$

Therefore, it is enough to prove this formula when R is a local ring with the maximal ideal $\mathfrak{m}$ and $f = R/\mathfrak{m}$ is the order of the residue field. As explained after Definition 2.3, there exists an element $e \in R \setminus \{0\}$ such that Re is contained in all non-zero ideals in R. If $g \in R$ such that $g \neq 0$ and g is not associated to e then $\mathrm{Ann}_R(g)$ is a proper sub-ideal of $\mathfrak{m}$. Consequently $\mu(R/\mathrm{Ann}_R(g)) = 0$ and hence $c(g, R) = 0$. If $g = 0$ then

$$c(0, R) = \varphi(R) = |R| \left(1 - \frac{1}{f}\right).$$

If g is associated with e (there are exactly $f - 1$ such elements) then by the short exact sequence

$$1 \to 1 + \mathfrak{m} \to R^\times \to (R/\mathfrak{m})^\times \to 1,$$

we have

$$c(g, R) = c(e, R) = -\frac{\varphi(R)}{\varphi(R/\mathfrak{m})} = -|\mathfrak{m}| = \frac{-|R|}{f}$$

We conclude that

$$M_k(R) = |R|^k \left[\left(1 - \frac{1}{f}\right)^k + (-1)^k \frac{f - 1}{f^k} \right]$$

□

We show that the graph-theory method can be used to provide another proof for the derivation of the third moment $M_3(R)$ of Ramanujan sums.

Proposition 2.10. *Let $R = \prod_{i=1}^{d} R_i$ is a product of finite local Frobenius rings. For each $1 \le i \le d$, let f_i be the order of the residue field of R_i. Then*

$$M_3(R) = \sum_{g\in R} c(g,R)^3 = |R|^3 \prod_{i=1}^{d}\left(1-\frac{1}{f_i}\right)\left(1-\frac{2}{f_i}\right).$$

Proof. We know that $M_3(R)$ is precisely the number of closed walks of length 3 in G_R. By [1, Proposition 2.3], for two fixed vertices $a, b \in G_R$ which are adjacent, the number of vertices c which are adjacent to both of them is

$$|R|\prod_{i=1}^{d}\left(1-\frac{2}{f_i}\right).$$

Therefore, the number of closed walks of length 3 is

$$M_3(R) = \varphi(R)|R| \times |R|\prod_{i=1}^{d}\left(1-\frac{2}{f_i}\right) = |R|^3\prod_{i=1}^{d}\left(1-\frac{1}{f_i}\right)\left(1-\frac{2}{f_i}\right).$$

□

We now discuss the second orthogonality property.

Theorem 2.11. *Let $r_1, r_2 \in R$ such that r_1 and r_2 are not associates. Then*

$$\sum_{g\in R} c(r_1 g, R)c(r_2 g, R) = 0.$$

We first give a proof using supercharacter theory.

Proof. For each $r \in R$

$$c(gr,R) = \sum_{a\in R^\times} \chi_{gr}(a) = \sum_{a\in R^\times}\chi_g(ra) = \frac{\varphi(R)}{|\mathrm{Stab}(r)|}\Omega_r(X_g).$$

Here, for each $r \in R$, $\mathrm{Stab}(r)$ is the stabilizer of r. Using this formula, Eq. (2.8), and the fact that $K_{r_1} \neq K_{r_2}$, we then see that

$$\begin{aligned}\sum_{g\in R} c(r_1 g, R)c(r_2 g) &= \sum_{\ell=1}^{m}\left[\sum_{g\in K_i} c(r_1 g, R)c(r_2 g)\right] \\ &= \frac{\varphi(R)}{|\mathrm{Stab}(r_1)|}\frac{\varphi(R)}{|\mathrm{Stab}(r_2)|}\sum_{\ell=1}^{m}|X_\ell|\Omega_{r_1}(X_\ell)\Omega_{r_2}(X_\ell) = 0.\end{aligned}$$

□

Similar to Theorem 2.6, we can also give a more direct proof of Theorem 2.11.

Proof.

$$\sum_{g\in R} c(r_1 g, R)c(r_2 g, R) = \sum_{g\in R}\left(\sum_{a\in R^\times}\chi_{gr_1}(a)\right)\left(\sum_{b\in R^\times}\chi_{r_2 g}(b)\right)$$
$$= \sum_{a\in R^\times}\sum_{b\in R^\times}\sum_{g\in R}\chi(g(r_1a+r_2b)) = \sum_{a\in R^\times}\sum_{b\in R^\times}\sum_{g\in R}\chi_{r_1a+r_2b}(g).$$

The inner sum is 0 when $r_1a + r_2b \neq 0$, but because $Rr_1 \neq Rr_2$, then $r_1a + r_2b \neq 0$ for any $a, b \in R^\times$. The proof follows. □

We now use Theorem 2.6 and Theorem 2.11 to explain some well-know orthogonality relations for classical Ramanujan sums. For $m, n \in \mathbb{Z}$, we note that the classical Ramanujan sum $c_n(m)$ is precisely $c(m, \mathbb{Z}/n)$.

Corollary 2.12. *We have the following orthogonality relations*

(1) *Suppose that n is a multiple of k. Then*

$$\sum_{m=1}^{n} c_k(m)^2 = \varphi(k)n.$$

(2) *Let p, q be two distinct numbers. Then*

$$\sum_{m=1}^{pq} c_p(m)c_q(m) = 0.$$

Proof. Let us prove the first statement. By definition, $c_k(m)$ only depends on m modulo k. Therefore

$$\sum_{m=1}^{n} c_k(m)^2 = \frac{n}{k}\sum_{m=1}^{k} c_k(m)^2 = \frac{n}{k}\sum_{m\in\mathbb{Z}/k} c(m, \mathbb{Z}/k)^2.$$

By Theorem 2.6 we know that

$$\sum_{m\in\mathbb{Z}/k} c(m, \mathbb{Z}/k)^2 = k\varphi(k).$$

This shows that

$$\sum_{m=1}^{n} c_k(m)^2 = n\varphi(k).$$

We now prove the second orthogonality relation. We remark that

$$c_p(m) = \sum_{\substack{1\le j\le p\\ \gcd(j,p)=1}} \zeta_p^{mj} = \frac{\varphi(p)}{\varphi(pq)}\sum_{\substack{1\le j\le pq\\ \gcd(j,pq)=1}} \zeta_p^{mj} = \frac{\varphi(p)}{\varphi(pq)}\sum_{\substack{1\le j\le pq\\ \gcd(j,pq)=1}} \zeta_{pq}^{mqj} = \frac{\varphi(p)}{\varphi(pq)}c(mq, \mathbb{Z}/pq).$$

Consequently

$$\sum_{m=1}^{pq} c_p(m)c_q(m) = \frac{\varphi(p)\varphi(q)}{\varphi(pq)^2}\sum_{m\in\mathbb{Z}/pq} c(mq, \mathbb{Z}/pq)c(mp, \mathbb{Z}/pq).$$

Since $p \neq q$, p and q are non-associate elements in $\mathbb{Z}/pq$. As a result, the above sum is 0 by Theorem 2.11.

□

3. RAMANUJAN DETERMINANT

Let R be a finite commutative Frobenius ring. We will again denote by $(\mathcal{K}, \mathcal{X})$ the supercharacter theory on $(R, +)$ associated with $R^\times$ as explained in Proposition 2.5. For each $1 \leq i \leq m$, fix a representative x_i of K_i. We first have the following theorem which gives an explicit description of the associated supercharacter table.

Theorem 3.1. *Let S be the supercharacter table associated with the pair $(\mathcal{K}, \mathcal{X})$. Then $S = C_R$ where*

$$C_R = [c(x_j, R/\text{Ann}_R(x_i))]_{1 \leq i,j \leq m}.$$

where

$$c(x_j, R/\text{Ann}_R(x_i)) = \frac{\varphi(R/\text{Ann}_R(x_i))}{\varphi(R/\text{Ann}_R(x_i x_j))}\mu(R/\text{Ann}_R(x_i x_j)). \tag{3.1}$$

Proof. By definition we have

$$\begin{aligned}
S_{ij} = \sigma_i(K_j) &= \sum_{\chi \in X_i} \chi(x_j) = \sum_{x \in K_i} \chi_x(x_j) = \sum_{x \in K_i} \chi_{x_j}(x) \\
&= \frac{|K_i|}{\varphi(R)} \sum_{u \in R^\times} \chi_{x_j}(g x_i) = \frac{\varphi(R/\text{Ann}_R(x_i))}{\varphi(R)} \sum_{u \in R^\times} \chi_{x_j x_i}(u) \\
&= \frac{\varphi(R/\text{Ann}_R(x_i))}{\varphi(R)} c(x_i x_j, R) = \frac{\varphi(R/\text{Ann}_R(x_i))}{\varphi(R)} \frac{\varphi(R)}{\varphi(R/\text{Ann}_{x_i x_j}(R))}\mu(R/\text{Ann}_{x_i x_j}(R)) \\
&= \frac{\varphi(R/\text{Ann}_R(x_i))}{\varphi(R/\text{Ann}_R(x_i x_j))}\mu(R/\text{Ann}_R(x_i x_j)).
\end{aligned}$$

□

By Theorem 3.2, we also have the following which generalizes [21, Theorem 1] and [14, Proposition 2.4].

Theorem 3.2. *Let R be a Frobenius ring. Then $|\det(C_R)| = |\det(S)| = |R|^{\frac{\tau(R)}{2}}$.*

We remark that while the definition of C_R depends on the theory of Ramanujan sums, the final formula does not. In other words, for a finite commutative ring, it makes perfect sense to define C_R as

$$C_R = \left[\frac{\varphi(R/\text{Ann}_R(x_i))}{\varphi(R/\text{Ann}_R(x_i x_j))}\mu(R/\text{Ann}_R(x_i x_j))\right]_{1 \leq i,j \leq m}.$$

Here $\{x_1, \ldots, x_m\}$ is a complete set of representatives for the cosets $R^\times \backslash R$; namely, they are a pairwise non-associates in R. We have the following proposition, which gives a new criterion for a finite commutative ring to be Frobenius.

Theorem 3.3. $\det(C_R) \neq 0$ *if and only if R is a finite commutative ring.*

Proof. We know that φ and μ are multiplicative with respect to direct product; namely if $R = R_1 \times R_2$ then

$$\varphi(R) = \varphi(R_1)\varphi(R_2), \mu(R) = \mu(R_1)\mu(R_2).$$

Therefore, upto an ordering, $C_R = C_{R_1} \otimes C_{R_2}$. Consequently,

$$\det(C_R) = \det(C_{R_1})^{\tau(R_2)} \det(C_{R_2})^{\tau(R_1)}. \tag{3.2}$$

By the Artin-Wedderburn theorem, a finite commutative ring is Frobenius if and only if it is a product of local finite commutative Frobenius ring. By Eq. (3.2), it is enough to prove that this proposition is true for the case R is local. If R is Frobenius, then Theorem 3.2 implies that $\det(C_R) \neq 0$. Let us assume that R is not Frobenius. By [9, Theorem 1], R have two distinct minimal ideals Re_1 and Re_2. The minimality condition implies that $\mathrm{Ann}_R(e_1) = \mathrm{Ann}_R(e_2) = \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of R. We claim that for each $r \in R$

$$\frac{\varphi(R/\mathrm{Ann}_R(r))}{\varphi(R/\mathrm{Ann}_R(e_1 r))}\mu(R/\mathrm{Ann}_R(e_1 r)) = \frac{\varphi(R/\mathrm{Ann}_R(r))}{\varphi(R/\mathrm{Ann}_R(e_2 r))}\mu(R/\mathrm{Ann}_R(e_2 r)).$$

This will, of course, imply that C_R is singular and hence $\det(C_R) = 0$. To prove this fact, we consider two cases. If $r \in \mathfrak{m}$, then $re_1 = re_2 = 0$. Therefore, both numbers are equal to $\varphi(R/\mathrm{Ann}_R(r))$. On the other hand, if $r \in R \setminus \mathfrak{m} = R^\times$, then $\mathrm{Ann}_R(e_1 r) = \mathrm{Ann}_R(e_2 r) = \mathfrak{m}$ and therefore these numbers are both equal to $-\frac{\varphi(R/\mathrm{Ann}_R(r))}{\varphi(R/\mathfrak{m})}$. $\square$

Question 3.4. What can we say about the rank of C_R?

4. Kluyver's formula

In this section, we discuss an equivalent definition of Ramanujan sums, often called the Kluyver formula in the literature. Various works have studied special cases of this formula. For example, [24] investigates the case where R is a quotient of the polynomial ring $\mathbb{F}_q[x]$, while [23] examines the case where R is a finite quotient of a Dedekind domain. The fact that these finite rings are Frobenius is proved in [16, Theorem 3.8, Theorem 3.9]. In this context, our theorem below provides a unified approach to Kluyver's formula.

Theorem 4.1. *Let R be a Frobenius ring and $g \in R$. Then*

$$c(g, R) = \sum_{Rg \subset I} N(I)\mu(R/\mathrm{Ann}_R(I)).$$

Here $N(I)$ is the order of the quotient ring R/I.

Proof. We remark that both sides of the equations are multiplicative with respect to the direct product. As a result, we only need to prove this statement when R is a local ring.

Since R is a local Frobenius ring, it has a unique minimal ideal namely $I_0 = Re$ for some $e \in R$. Furthermore, $\mathrm{Ann}_R(I_0) = \mathrm{Ann}_R(e) = \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of R. We note that since R is Frobenius, it has an elegant duality property: for each ideal I in R, $\mathrm{Ann}_R(\mathrm{Ann}_R(I)) = I$ (see [9]). Let us consider the right hand side. By definition, $\mu(R/\mathrm{Ann}_R(I)) = 0$ unless $\mathrm{Ann}_R(I) = R$ (when $I = 0$) or $\mathrm{Ann}_R(I) = \mathfrak{m}$ (when $I = Re$). We consider a few cases.

Case 1. $g = 0$. In this case, we have $c(g, R) = \varphi(R)$. On the other hand, the right hand side is equal to

$$|R| - |R/Re| = |R| - \frac{|R|}{|Re|} = |R| - \frac{|R|}{|R/\mathfrak{m}|} = |R| - |\mathfrak{m}| = \varphi(R) = c(g, R).$$

Case 2. g is associated with e; namely $Rg = Re$. In this case, by Eq. (2.10), $c(g, R) = -\frac{\varphi(R)}{\varphi(R/\mathfrak{m})}$ which is equal to $-|m|$ by the short exact sequence

$$1 \to 1 + \mathfrak{m} \to R^\times \to (R/\mathfrak{m})^\times \to 1.$$

On the hand, the only non-zero terms on the right side occur at $I = Re$. Therefore, the right hand side is equal to to $-|R/Re| = -|\mathfrak{m}|$.

Case 3. $g \neq 0$ and $Rg \neq Re$. In this case, both sides are equal to 0. □

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