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On the arithmetic of join rings over finite fields [☆]Sunil K. Chebolu ^{a,*}, Jonathan L. Merzel ^b, Ján Mináč ^c,
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ABSTRACT

In this paper we consider some interesting and surprising interactions of several topics including representation theory, matrix algebra, and number theory. Given a collection $\{G_i\}_{i=1}^d$ of finite groups and a ring R , we have previously introduced and studied certain foundational properties of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$. This ring bridges two extreme worlds: matrix rings $M_n(R)$ on one end and group rings $R[G]$ on the other. The construction of this ring was motivated by various problems in graph theory, network theory, nonlinear dynamics, and neuroscience. In this largely self-contained paper, we continue our investigations of this ring, focusing more on its arithmetic properties. We begin by constructing

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a generalized augmentation map that gives a structural decomposition of this ring. This decomposition allows us to compute the zeta function of the join of group rings. We show that the join of group rings is a natural home for studying the concept of simultaneous primitive roots for a given set of primes. This concept is related to the order of the unit group of the join of group rings. Finally, we characterize the join of group rings over finite fields with the property that the order of every unit divides a fixed number. Remarkably, Mersenne and Fermat primes unexpectedly emerge within the context of this exploration.

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1. Introduction

Let G be a finite group. The concept of G -circulant matrices, defined in 2.1, has a rich mathematical history. Dedekind initially introduced these matrices during his study of normal bases for Galois extensions. His focus was on understanding the factorizations of the determinants of G -circulant matrices. While his success was notable in cases where G is abelian, his progress in the non-abelian realm was limited. This work led to correspondence with Frobenius in 1896. Subsequently, Frobenius made a pivotal discovery. He showed that the determinant of a generic G -circulant matrix decomposes into a product of irreducible factors over the field of complex numbers corresponding to the linear irreducible representations of the group G . In particular, when G is a cyclic group, we have an explicit description of the spectrum of G -circulant matrices. This description is often referred to as the Circulant Diagonalization Theorem in the literature (see [9] for an extensive treatment of this topic). Due to their elegance and explicit nature, circulant matrices have found applications in many scientific fields, such as spectral graph theory, coding theory, neuroscience, and nonlinear dynamics (see [1,2,9,10,12,18,19,26,32]). In

particular, in [2], using the spectral decomposition of a circulant network, we are able to explain various traveling wave patterns in networks of phase oscillators.

In [8], we introduce a natural generalization of G -circulant matrices. More precisely, given a collection of finite groups G_1, G_2, \dots, G_d and a ring R , we introduce the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$ (see Section 2.2 for the precise definition of this ring). When $d = 1$, the ring $\mathcal{J}_G(R)$ is exactly the ring of all G -circulant matrices with entries in R . Furthermore, $\mathcal{J}_G(R)$ is naturally isomorphic to the group ring $R[G]$. We also remark that when all G_i are the trivial group, the join ring is naturally isomorphic to $M_d(R)$, the ring of all square matrices of size $d \times d$ with coefficients in R . The introduction of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$ is motivated by a construction in graph theory known as the joined union of graphs, and by a desire to understand nonlinear dynamics in multilayer networks of oscillators (see [10,24,26,27]). In [8], we discuss some fundamental ring-theoretic properties of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$ such as its center, its semisimplicity, its Jacobson radical, the structure of its unit group, and much more. In this article, we discuss some further properties of this ring, focusing on the case that R is a finite field. This article presents our continuing effort to develop a systematic understanding of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$. We have made a concerted effort to ensure that our work is accessible to a broad readership. To this end, we provide a self-contained review of the fundamental notions and key results required for a complete understanding of the text.

We now summarize our main results. The definitions of the join of group rings and the associated zeta functions can be found in Section 2 and Section 3, respectively. Our first result is a structural decomposition of the join rings.

Theorem 1.1 (*Decomposition of Join Rings*). *Let G_1, \dots, G_d be finite groups and $H_i \trianglelefteq G_i$ such that $|H_i|$ is invertible in a unital ring R . Then, there exists a ring isomorphism*

$$J_{G_1, \dots, G_d}(R) \cong J_{G_1/H_1, \dots, G_d/H_d}(R) \times \prod_{i=1}^d \Delta_R(G_i, H_i),$$

where $\Delta_R(G_i, H_i)$ is the kernel of the augmentation map $R[G_i] \rightarrow R[G_i/H_i]$.

Note that, in the special case when all the $|G_i|$'s are invertible in R , we get

$$\mathcal{J}_{G_1, G_2, \dots, G_d}(R) \cong M_d(R) \times \prod_{i=1}^d \Delta_R(G_i).$$

The above structural decomposition helps compute the zeta function for the join of group rings. We refer the reader to Section 2 for the definition of these rings and to Section 3 for the definition of their zeta functions.

Theorem 1.2 (*Zeta Function of Join Rings*). *Let \mathbb{F}_q be a finite field, and suppose that $|G_i|$ is invertible in \mathbb{F}_q for all $1 \leq i \leq d$. Then the zeta function of the join ring satisfies*

$$\zeta_{J_{G_1, \dots, G_d}(\mathbb{F}_q)}(s) = (1 - q^{-s})^{d-1} \prod_{i=1}^d \zeta_{\mathbb{F}_q[G_i]}(s).$$

For the general case when some $|G_i|$ is not invertible in \mathbb{F}_q , we refer the reader to Theorem 3.15. We use these results to explicitly compute the zeta function in a number of examples; see Corollary 3.9 and various other examples in Section 3.

We then use these zeta functions to study q -rooted primes. A prime p is said to be q -rooted if q is a primitive root modulo p . 2-rooted primes and their characterizations were studied in [6]. In the current paper, we extend those results to the odd primary case and also study simultaneous q -rooted primes in the framework of zeta functions.

Theorem 1.3 (Characterization of q -Rooted Primes). *Let q, p_1, \dots, p_d be prime numbers with $p_i \neq q$ for all i . The following are equivalent:*

- (1) Each p_i is a q -rooted prime (i.e., q is a primitive root modulo p_i).
- (2) The order of the pole at $s = 0$ of $\zeta_{J_{\mathbb{Z}/p_1, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)}(s)$ is $d + 1$.
- (3) The order of the unit group of $J_{\mathbb{Z}/p_1, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$ is

$$\prod_{i=1}^d (q^{p_i-1} - 1) \cdot \prod_{i=0}^{d-1} (q^d - q^i).$$

We end the paper with an investigation of Δ_{p^r} -rings. A ring is said to be a Δ_{p^r} -ring if $u^{p^r} = 1$ for all units u in the ring. These rings were introduced in [6], where the following question was raised: When is the group algebra kG a Δ_p -ring? In [6], the authors addressed this for the case when G is an abelian group and $r = 1$. Here we extend those results to all finite groups and $r \geq 1$ 5.13, and also to join algebras defined over a finite field.

Theorem 1.4 (Classification of Join Rings that are Δ_{p^r} -Rings). *Let $d \geq 2$. Then the join ring $J_{G_1, \dots, G_d}(\mathbb{F}_q)$ is a Δ_{p^r} -ring if and only if:*

- (1) $p = q = 2$,
- (2) Each G_i is a 2-group,
- (3) At most one G_i is trivial.
- (4) $2^r \geq \max_{1 \leq i \leq d} \exp(U_1(\mathbb{F}_2[G_i]))$, where $U_1(\mathbb{F}_2[G_i])$ is the set of normalized units in $\mathbb{F}_2[G_i]$.

1.1. Outline

The structure of this article is as follows. In Section 2, we study some further ring-theoretic properties of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$. Among various things that we discover, we discuss a natural construction of the generalized augmentation map (a

special case of this construction is previously discussed in [8]). Section 3 studies the zeta functions of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$ when $R = \mathbb{F}_q$ is a finite field. More precisely, we describe how to explicitly calculate the zeta function of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ in terms of the zeta functions of $\mathcal{J}_{G_i}(\mathbb{F}_q)$. In Section 4, we discuss the order of the unit group of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ and explain its connection with Artin’s conjecture on primitive roots. For instance, we find several equivalent conditions for when a given prime q is simultaneously a primitive root for a set of primes $\{p_i\}$. These equivalent conditions are based on the cardinality of the unit group and the order of the pole of the zeta function for the join ring $\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$. Finally, in Section 5, we classify all join rings $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ that have the property that every unit u in them satisfies $u^{p^r} = 1$, where p is a prime number and r is a positive integer. Such rings are called Δ_{p^r} rings and they are well-studied in the literature; see [4,6,7]. In particular, [6] focuses on the aforementioned property when $d = 1$, $r = 1$, and the finite group involved is abelian. The results of Section 5 advance beyond these parameters, broadening the outcome to encompass all finite groups and positive integer values of d and r .

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2. Some ring-theoretic properties of $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$

2.1. The ring of G -circulant matrices

Let $G = \{g_1 (= e), g_2, \dots, g_n\}$ be a finite group of order n (note that we have fixed an ordering on G). We first recall the definition of a G -circulant matrix (for more details, see [8,16,18]).

Definition 2.1. An $n \times n$ G -circulant matrix over R is an $n \times n$ matrix

$$A = \begin{bmatrix} a_{g_1, g_1} & a_{g_1, g_2} & \cdots & a_{g_1, g_n} \\ a_{g_2, g_1} & a_{g_2, g_2} & \cdots & a_{g_2, g_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g_n, g_1} & a_{g_n, g_2} & \cdots & a_{g_n, g_n} \end{bmatrix}$$

over R with the property that for all $g, g_i, g_j \in G$, $a_{g_i, g_j} = a_{gg_i, gg_j}$.

We remark that a G -circulant matrix A is completely determined by its first row and the multiplication table of G , as we must have $a_{g_i, g_j} = a_{g_1, g_i^{-1}g_j}$. For simplicity, we

sometimes write $A = \text{circ}([a_g]_{g \in G})$ (where we define the doubly indexed quantity a_{g_i, g_j} by $a_{g_i^{-1} g_j}$). Let $\mathcal{J}_G(R)$ be the set of all G -circulant matrices over R and

$$R[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in R \right\},$$

the group ring of G with coefficients in R . In [8], we reproved the following theorem of Hurley.

Proposition 2.2. (Hurley) *The map $\alpha : R[G] \rightarrow \mathcal{J}_G(R)$ sending*

$$\sum_{g \in G} a_g g \mapsto \text{circ}([a_g]_{g \in G}),$$

is a ring isomorphism. In particular, under this isomorphism, units in the group ring correspond to invertible G -circulant matrices.

2.2. The join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$

We recall the definition of the join matrix (see [8, Definition 3.1]).

Definition 2.3. Let R be a (unital, associative) ring, G_1, \dots, G_d finite groups of respective orders k_1, \dots, k_d , and let C_i be G_i -circulant ($1 \leq i \leq d$) over R . By a join of C_1, \dots, C_d over R , we mean a matrix of the form

$$A = \begin{bmatrix} C_1 & a_{12} J_{k_1, k_2} & \cdots & a_{1d} J_{k_1, k_d} \\ a_{21} J_{k_2, k_1} & C_2 & \cdots & a_{2d} J_{k_2, k_d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} J_{k_d, k_1} & a_{d2} J_{k_d, k_2} & \cdots & C_d \end{bmatrix}, \tag{*}$$

where $a_{ij} \in R$ ($1 \leq i \neq j \leq d$) and $J_{r,s}$ denotes the $r \times s$ matrix, all of whose entries are $1 \in R$.

We remark that we came upon the concept of a join matrix through our work on multilayer networks of phase oscillators (see [10,24,27]). As in [8], we will denote by $\mathcal{J}_{G_1, \dots, G_d}(R)$, the set of all such joins as the C_i vary independently through all G_i -circulant matrices ($1 \leq i \leq d$) and the a_{ij} vary independently through all elements of R ($1 \leq i \neq j \leq d$). In [8], we showed the following.

Proposition 2.4. ([8, Section 3]) *$\mathcal{J}_{G_1, \dots, G_d}(R)$ has the structure of a unital ring. Furthermore, there is an augmentation map $\epsilon : \mathcal{J}_{G_1, \dots, G_d}(R) \rightarrow M_d(R)$ that generalizes the augmentation map on group rings.*

Since we identified $\mathcal{J}_{G_1, \dots, G_d}(R)$ as a subring of a matrix ring over R , it is clear that in the case when R is a field k , $\mathcal{J}_{G_1, \dots, G_d}(k)$ has the structure of a k -algebra.

2.3. The generalized augmentation map

Let G be a finite group and H be a normal subgroup of G . Then, there is a canonical ring map known as the augmentation map

$$\epsilon : R[G] \rightarrow R[G/H], \tag{2.1}$$

which extends the quotient map $G \rightarrow G/H$ that sends $g \mapsto \bar{g}$. When $H = G$, this is exactly the standard augmentation map $\epsilon : R[G] \rightarrow R$ mentioned in the previous section. More concretely, this augmentation map is defined by

$$\epsilon \left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g.$$

In this section, we show that there is a natural analog of this augmentation map in the setting of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$. More precisely, let G_i be a finite group and H_i a normal subgroup of G_i for all $1 \leq i \leq d$. Suppose that the orders of $G_i, H_i, G_i/H_i$ are k_i, r_i, s_i respectively (so $k_i = r_i s_i$). Let us consider the following map

$$\epsilon : \mathcal{J}_{G_1, G_2, \dots, G_d}(R) \rightarrow \mathcal{J}_{G_1/H_1, G_2/H_2, \dots, G_d/H_d}(R), \tag{2.2}$$

defined by

$$\begin{aligned} & \begin{bmatrix} C_1 & a_{12}J_{k_1, k_2} & \cdots & a_{1d}J_{k_1, k_d} \\ a_{21}J_{k_2, k_1} & C_2 & \cdots & a_{2d}J_{k_2, k_d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}J_{k_d, k_1} & a_{d2}J_{k_d, k_2} & \cdots & C_d \end{bmatrix} \\ & \mapsto \begin{bmatrix} \epsilon(C_1) & r_2 a_{12}J_{s_1, s_2} & \cdots & r_d a_{1d}J_{s_1, s_d} \\ r_1 a_{21}J_{s_2, s_1} & \epsilon(C_2) & \cdots & r_d a_{2d}J_{s_2, s_d} \\ \vdots & \vdots & \ddots & \vdots \\ r_1 a_{d1}J_{s_d, s_1} & r_2 a_{d2}J_{s_d, s_2} & \cdots & \epsilon(C_d). \end{bmatrix} \end{aligned}$$

Here ϵ is the classical augmentation map $R[G_i] \rightarrow R[G_i/H_i]$ as defined in Equation (2.1). We remark that the row and column sum of a G -circulant matrix $A = \text{circ}([a_g]_{g \in G})$ are all equal to $\sum_{g \in G} a_g$. This type of matrix has a special name which we now recall.

Definition 2.5. (see [25]) Let R be a ring. A matrix $A \in M_n(R)$ is called a semimagic square if its row and column sums are equal; i.e., there exists a constant $\sigma(A)$ such that

$$\sum_{i=1}^n a_{ij} = \sum_{j=1}^n a_{ij} = \sigma(A).$$

We have the following.

Proposition 2.6. *The map $\epsilon : \mathcal{J}_{G_1, G_2, \dots, G_d}(R) \rightarrow \mathcal{J}_{G_1/H_1, G_2/H_2, \dots, G_d/H_d}(R)$ is a ring homomorphism.*

Proof. This follows from direct calculations. Two key identities are the following.

- (1) $J_{m,n} \times J_{n,p} = nJ_{m,p}$.
- (2) $AJ_{m,n} = \epsilon(A)J_{m,n}$ where A is a semimagic square of size $m \times m$ and $\epsilon(A)$ is the row sum of A . Similarly $J_{m,n}B = \epsilon(B)J_{m,n}$ if B is a semimagic square of size $n \times n$. \square

Given a group homomorphism $\varphi : G \rightarrow G'$ we have functorially a map $\Phi^{G,G'} : R[G] \rightarrow R[G']$. (We could, of course, also permit a homomorphism from R to another ring R' .) In the case where $H \triangleleft G$ and φ is the canonical map $\pi_{G,G/H} : G \rightarrow G/H$, the map $\Phi^{G,G/H}$ is what we above called the augmentation map ϵ , which we will temporarily designate $\epsilon^{G,G/H}$. By functoriality, we mean that if we also have $\varphi' : G' \rightarrow G''$ then $\Phi^{G,G''} = \Phi^{G',G''} \circ \Phi^{G,G'}$. We therefore have

Lemma 2.7. (a) *If $\varphi : G \rightarrow G'$ is a homomorphism and if $H \triangleleft G$ with $\varphi(H) < H' \triangleleft G'$, then we have a commutative diagram*

$$\begin{array}{ccc}
 R[G] & \xrightarrow{\epsilon^{G,G/H}} & R[G/H] \\
 \Phi^{G,G'} \downarrow & & \downarrow \Phi^{G/H,G'/H'} \\
 R[G'] & \xrightarrow{\epsilon^{G',G'/H'}} & R[G'/H']
 \end{array}$$

(b) *If $H, K \triangleleft G$ with $H < K$ then $\epsilon^{G,G/K} = \epsilon^{G/H,G/K} \circ \epsilon^{G,G/H}$.*

Proof. (a) As $\epsilon^{G,G/H} = \Phi^{G,G/H}$ and $\epsilon^{G',G'/H'} = \Phi^{G',G'/H'}$, by functoriality both compositions in the diagram are $\Phi^{G,G'/H'}$ induced by $\pi_{G',G'/H'} \circ \varphi$.

(b) Apply (a) to the case $G' = G, H' = K$. \square

We would like to generalize the above to join rings. Unfortunately, given homomorphisms $\varphi_i : G_i \rightarrow G'_i, i = 1, \dots, d$ there is in general no apparent naturally induced map $\Phi : \mathcal{J}_{G_1, \dots, G_d}(R) \rightarrow \mathcal{J}_{G'_1, \dots, G'_d}(R)$. The problem is that the natural image of $J_{k_i, k_i} \in R[G_i]$ is not a multiple of $J_{k'_i, k'_i} \in R[G'_i]$; however, this holds when the maps $\varphi_i : G_i \rightarrow G'_i$ are surjective. In that case, we are up to isomorphism back to the earlier situation of Proposition 2.6, and we will also denote the map ϵ of that proposition by $\epsilon^{\{G_i\}, \{G'_i\}}$. Corresponding to the lemma above, we now have

Proposition 2.8. (a) *Given surjective homomorphisms $\varphi_i : G_i \rightarrow G'_i, i = 1, \dots, d$ and normal subgroups $H_i \triangleleft G_i$ with $\varphi(H_i) < H'_i \triangleleft G'_i$, we then have a commutative diagram*

$$\begin{array}{ccc}
 J_{G_1, \dots, G_d}(R) & \xrightarrow{\epsilon^{\{G_i\}, \{G_i/H_i\}}} & J_{G_1/H_1, \dots, G_d/H_d}(R) \\
 \epsilon^{\{G_i\}, \{G'_i\}} \downarrow & & \downarrow \epsilon^{\{G_i/H_i\}, \{G'_i/H'_i\}} \\
 J_{G'_1, \dots, G'_d}(R) & \xrightarrow{\epsilon^{\{G'_i\}, \{G'_i/H'_i\}}} & J_{G'_1/H'_1, \dots, G'_d/H'_d}(R)
 \end{array}$$

(b) If $H_i, K_i \triangleleft G_i$ with $H_i < K_i, i = 1, \dots, d$ then

$$\epsilon^{\{G_i\}, \{G_i/K_i\}} = \epsilon^{\{G_i/H_i\}, \{G_i/K_i\}} \circ \epsilon^{\{G_i\}, \{G_i/H_i\}}.$$

Proof. (a) For a matrix

$$\left(\begin{array}{cccc}
 C_1 & a_{12}J_{k_1, k_2} & \cdots & a_{1d}J_{k_1, k_d} \\
 a_{21}J_{k_2, k_1} & C_2 & \cdots & \alpha_{2d}J_{k_2, k_d} \\
 \vdots & \vdots & \ddots & \vdots \\
 \alpha_{d1}J_{k_d, k_1} & \alpha_{d2}J_{k_d, k_2} & \cdots & C_d
 \end{array} \right) \in J_{G_1, \dots, G_d}(R)$$

we consider the image of its blocks under the compositions $\epsilon^{\{G_i/H_i\}, \{G'_i/H'_i\}} \circ \epsilon^{\{G_i\}, \{G_i/H_i\}}$ and $\epsilon^{\{G'_i\}, \{G'_i/H'_i\}} \circ \epsilon^{\{G_i\}, \{G'_i\}}$. For the blocks on the diagonal, these images coincide by part (a) of the above lemma. The i, j block with $i \neq j$ accumulates factors of $|H_j|$ and $|(G_j/H_j)/(G'_j/H'_j)|$ under the composition $\epsilon^{\{G_i/H_i\}, \{G'_i/H'_i\}} \circ \epsilon^{\{G_i\}, \{G_i/H_i\}}$ and factors of $|G_j|/|G'_j|$ and $|H'_j|$ under the composition $\epsilon^{\{G'_i\}, \{G'_i/H'_i\}} \circ \epsilon^{\{G_i\}, \{G'_i\}}$, so the image either way is $\frac{|G_j||H'_j|}{|G'_j|} a_{ij} J_{G'_i/H'_i, G'_j/H'_j}$.

(b) Apply part (a) to the case $G'_i = G_i, H'_i = K_i$. \square

2.4. A decomposition of $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$

Let $\Delta_R(G, H) := \ker(R[G] \rightarrow R[G/H])$ be the kernel of the augmentation map as defined in Equation (2.1) (when R is clear from the context, we will simply write $\Delta(G, H)$.) Suppose further that $|H|$ is invertible in R . Let

$$e_H = \frac{1}{|H|} \sum_{h \in H} h.$$

It can be shown that e_H is a central idempotent in $R[G]$; see also [23, Lemma 3.6.6]. Furthermore, by [23, Proposition 3.6.7], we have

Proposition 2.9. *We have a direct product of rings*

$$R[G] \cong R[G]e_H \times R[G](1 - e_H).$$

Furthermore

$$R[G]e_H \cong R[G/H],$$

and

$$R[G](1 - e_H) = \Delta_R(G, H).$$

Corollary 2.10. (see [23, Corollary 3.6.9]) Suppose that $|G|$ is invertible in R . Let $\Delta_R(G)$ be the augmentation ideal. Then

$$R[G] \cong R \times \Delta_R(G).$$

We can generalize this proposition to the join ring as follows.

Theorem 2.11. Let G_1, \dots, G_d be finite groups. For $1 \leq i \leq d$, let H_i be a normal subgroup such that $|H_i|$ is invertible in R . Then, there exists an isomorphism

$$\mathcal{J}_{G_1, \dots, G_d}(R) \cong \mathcal{J}_{G_1/H_1, \dots, G_d/H_d}(R) \times \prod_{i=1}^d \Delta_R(G_i, H_i).$$

Proof. Let $f_i = f_{G_i} = 1 - e_{H_i} \in R[G_i]$ where e_{H_i} is defined as above. Since the ring of all G_i -circulant matrices is isomorphic to the group ring $R[G_i]$, we can also consider f_i as a G_i -circulant matrix. Let \tilde{f}_i be the following matrix in $\mathcal{J}_{G_1, G_2, \dots, G_d}(R)$

$$\tilde{f}_i = \left[\begin{array}{c|c|c|c} 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ \hline \vdots & \vdots & f_i & \vdots \\ \hline 0 & 0 & \dots & 0 \end{array} \right].$$

In other words, all blocks of \tilde{f}_i , except the i -diagonal block which is f_i , are 0. Additionally, we define

$$\tilde{f}_{d+1} = I_n - \sum_{i=1}^d \tilde{f}_i = \bigoplus_{i=1}^d e_{G_i}.$$

Then we have the following ring isomorphism

$$\mathcal{J}_{G_1, \dots, G_d}(R) \cong \tilde{f}_{d+1} \mathcal{J}_{G_1, \dots, G_d}(R) \times \prod_{i=1}^d \tilde{f}_i \mathcal{J}_{G_1, \dots, G_d}(R).$$

We can see that for $1 \leq i \leq d$

$$\tilde{f}_i \mathcal{J}_{G_1, \dots, G_d}(R) \cong \Delta_R(G_i, H_i).$$

Additionally, the augmentation map

$$\epsilon : \mathcal{J}_{G_1, G_2, \dots, G_d}(R) \rightarrow \mathcal{J}_{G_1/H_1, \dots, G_d/H_d}(R)$$

induces a ring isomorphism

$$\epsilon : \tilde{f}_{d+1}\mathcal{J}_{G_1, G_2, \dots, G_d}(R) \rightarrow \mathcal{J}_{G_1/H_1, \dots, G_d/H_d}(R). \quad \square$$

Here is a direct corollary of this theorem.

Corollary 2.12. (See also [8, Theorem 3.16]) Suppose that $|G_i|$ are invertible in R . Then

$$\mathcal{J}_{G_1, G_2, \dots, G_d}(R) \cong M_d(R) \times \prod_{i=1}^d \Delta_R(G_i).$$

3. Zeta function of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$

Let \mathbb{F}_q be the finite field with $q = p^r$ elements where p is a prime number. In this section, we study the zeta function of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$. We first recall the definition of the zeta function of a finite-dimensional \mathbb{F}_q -algebra as defined in [13].

First, consider the case where R is a commutative finite dimensional \mathbb{F}_q -algebra. The Hasse-Weil zeta function of R is defined to be

$$\zeta_R(s) = \prod_{m \subset R} (1 - \#(R/m)^{-s})^{-1}, \tag{3.1}$$

where m runs over all maximal ideals of R (see [13]). As observed in [13], when R is not commutative, the Hasse-Weil zeta function of R can be defined as follows. (We refer readers to [20,21] for some further motivations for this definition.)

Definition 3.1. (see [13]) Let R be a finite-dimensional \mathbb{F}_q -algebra. The following Euler product gives the Hasse-Weil zeta function of R

$$\zeta_R(s) = \prod_M (1 - |\text{End}_R(M)|^{-s})^{-1}, \tag{3.2}$$

where M runs over the isomorphism classes of (finite) simple left R -modules.

We remark that since R is a finite ring, all simple left R -modules are automatically finite. Furthermore, by [13, Lemma 2.7.1], another equivalent definition of $\zeta_R(s)$ is

$$\zeta_R(s) = \prod_{m \in \mathfrak{P}(R)} (1 - N(\mathfrak{m})^{-s})^{-1}, \tag{3.3}$$

where $\mathfrak{P}(R)$ is the set of all two-sided ideals \mathfrak{m} in R such that R/\mathfrak{m} is isomorphic to a matrix ring $M_r(k)$ with k a finite extension of \mathbb{F}_q and $N(\mathfrak{m}) = |k|$.

For a finite-dimensional \mathbb{F}_q -algebra R , we denote

$$R^{\text{ss}} = R/\text{Rad}(R), \tag{3.4}$$

where $Rad(R)$ is the Jacobson radical of R . It is well-known that $Rad(R^{ss}) = 0$. Additionally since R is Artinian, R^{ss} is Artinian as well. Consequently, R^{ss} is a semisimple algebra. We have the following observation.

Proposition 3.2. *Let R be a finite dimensional \mathbb{F}_q algebra and $Rad(R)$ the Jacobson radical of R . Let $\mathfrak{m} \in \mathfrak{P}(R)$. Then*

- (1) $Rad(R) \subset \mathfrak{m}$.
- (2) The map $\mathfrak{m} \mapsto \bar{\mathfrak{m}} := \mathfrak{m}/Rad(R)$ from $\mathfrak{P}(R) \rightarrow \mathfrak{P}(R^{ss})$ is a bijection. Furthermore $N(\mathfrak{m}) = N(\bar{\mathfrak{m}})$.

Proof. By definition $R/\mathfrak{m} \cong M_r(k)$ for some $r \geq 1$ and a field k . The first statement hence follows from [29, Section 4.3, Lemma b]. The second statement then follows naturally from the first statement. \square

A direct consequence of this proposition is the following.

Proposition 3.3. *Suppose R is a finite-dimensional \mathbb{F}_q -algebra. Then*

$$\zeta_R(s) = \zeta_{R^{ss}}(s).$$

We investigate some further properties of the zeta function of a finite-dimensional \mathbb{F}_q -algebra.

Proposition 3.4. *Let R and T be two finite-dimensional \mathbb{F}_q -algebras. Then*

- (1) $\zeta_{R \times T}(s) = \zeta_R(s)\zeta_T(s)$.
- (2) If R and T are Morita equivalent, then $\zeta_R(s) = \zeta_T(s)$.

Proof. Part (1) follows directly from the definition of the zeta function. Part (2) is [13, Proposition 2.2]. \square

We discuss some concrete examples of R and their zeta functions.

Example 3.5. Let us consider $R = M_n(\mathbb{F}_q)$. Since $M_n(\mathbb{F}_q)$ is Morita equivalent to \mathbb{F}_q , Proposition 3.4 shows that

$$\zeta_{M_n(\mathbb{F}_q)}(s) = \zeta_{\mathbb{F}_q}(s) = (1 - q^{-s})^{-1}.$$

Example 3.6. Let G be a finite group such that $|G|$ is invertible in \mathbb{F}_q . Let $R = \mathbb{F}_q[G]$. Suppose further that G splits over \mathbb{F}_q ; i.e.,

$$\mathbb{F}_q[G] = \prod_{i=1}^d M_{n_i}(\mathbb{F}_q).$$

Then

$$\zeta_{\mathbb{F}_q[G]}(s) = \prod_{i=1}^d \zeta_{M_{n_i}(\mathbb{F}_q)}(s) = (1 - q^{-s})^{-d}.$$

In general, if G does not split over \mathbb{F}_q then the calculation of $\zeta_{\mathbb{F}_q[G]}(s)$ is less explicit. However, when G is abelian, we can explicitly describe the zeta function of $\mathbb{F}_q[G]$. Before we state the key theorem, we recall the following definition.

Definition 3.7. Let d be a positive integer and a an integer such that $\gcd(a, d) = 1$. The order of a with respect to d , denoted by $\text{ord}_d(a)$ is the smallest positive integer t such that $a^t \equiv 1 \pmod{d}$.

We are now ready to state the key theorem that allows us to compute the zeta function of $\mathbb{F}_q[G]$ where G is an abelian group.

Theorem 3.8. [23, Theorem 3.5.4] Let G be a finite abelian group of order n which is prime to q . Then

$$\mathbb{F}_q[G] \cong \bigoplus_{d|n} a_d \mathbb{F}_q[\zeta_d],$$

where ζ_d is a primitive root of unity of order d and $a_d = \frac{n_d}{[\mathbb{F}_q(\zeta_d) : \mathbb{F}_q]}$. Here n_d is the number of elements of order d in G . Note also that

$$[\mathbb{F}_q(\zeta_d) : \mathbb{F}_q] = \text{ord}_d(q).$$

Corollary 3.9. Let G be a finite abelian group of order n prime to q . Then

$$\zeta_{\mathbb{F}_q[G]}(s) = \prod_{d|n} (1 - q^{-\text{ord}_d(q)s})^{-a_d},$$

where a_d and $\text{ord}_d(q)$ are as above.

We also remark that in some special cases, the zeta function of $\zeta_{\mathbb{F}_q[G]}(s)$ in the modular case (namely $|G| = 0$ in \mathbb{F}_q) can be deduced from the semisimple case (namely when $|G|$ is invertible in \mathbb{F}_q). This is a consequence of Proposition 3.3 and the following theorem.

Theorem 3.10. ([28, Theorem 16.6]) Let G be a finite group. Suppose that H is a normal p -Sylow subgroup of G . Then, the Jacobson radical of $\mathbb{F}_q[G]$ is the kernel of the augmentation map

$$\epsilon : \mathbb{F}_q[G] \rightarrow \mathbb{F}_q[G/H].$$

Consequently, $\mathbb{F}_q[G]^{ss} \cong \mathbb{F}_q[G/H]$ and

$$\zeta_{\mathbb{F}_q[G]}(s) = \zeta_{\mathbb{F}_q[G/H]}(s).$$

We discuss another example of a class of rings where we can compute their zeta functions quite explicitly. Specifically, we can check that the set of all semimagic squares of size $n \times n$ as defined in Definition 2.5 is a subalgebra of $M_n(k)$. For simplicity, we will denote this ring by $SM_n(k)$. By [25], we can describe the semisimplification of $SM_n(k)$ explicitly.

Theorem 3.11. [25, Theorem 2, Theorem 3] *Let k be a field of characteristic $p \geq 0$. Then*

(1) *If $p \nmid n$ then*

$$SM_n(k) \cong k \times M_{n-1}(k).$$

(2) *If $p|n$ then the algebra $SM_n(k)$ is not semisimple. Its simplification is given by*

$$SM_n(k)^{ss} \cong k \times M_{n-2}(k).$$

Corollary 3.12. *Let $SM_n(\mathbb{F}_q)$ be the ring of all semimagic squares of size $n \times n$ over \mathbb{F}_q with $n \geq 1$. Then*

(1) *If $n = 1$ then $\zeta_{SM_1(\mathbb{F}_q)}(s) = (1 - q^{-s})^{-1}$.*

(2) *If $n = 2$ then*

$$\zeta_{SM_2(\mathbb{F}_q)}(s) = \begin{cases} (1 - q^{-s})^{-2} & \text{char}(\mathbb{F}_q) \neq 2 \\ (1 - q^{-s})^{-1} & \text{char}(\mathbb{F}_q) = 2. \end{cases}$$

(3) *If $n \geq 3$ then*

$$\zeta_{SM_n(\mathbb{F}_q)}(s) = (1 - q^{-s})^{-2}.$$

We now compute explicitly the zeta function of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ in terms of the zeta functions for $\mathbb{F}_q[G_i]$ for $1 \leq i \leq d$. We first consider the semisimple case, where all $|G_i|$ are invertible in \mathbb{F}_q (by [8, Corollary 5.3]). In this case, by Corollary 2.12, we have

$$\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q) \cong M_d(\mathbb{F}_q) \times \prod_{i=1}^d \Delta_{\mathbb{F}_q}(G).$$

Consequently

$$\zeta_{\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)}(s) = \zeta_{M_d(\mathbb{F}_q)}(s) \prod_{i=1}^d \zeta_{\Delta_{\mathbb{F}_q}(G_i)}(s) = (1 - q^{-s})^{-1} \prod_{i=1}^d \zeta_{\Delta_{\mathbb{F}_q}(G_i)}(s).$$

Furthermore, we also have

$$\mathbb{F}_q[G_i] \cong \mathbb{F}_q \times \Delta_{\mathbb{F}_q}(G_i),$$

and therefore

$$\zeta_{\mathbb{F}_q[G_i]}(s) = \zeta_{\mathbb{F}_q}(s) \zeta_{\Delta_{\mathbb{F}_q}(G_i)}(s) = (1 - q^{-s})^{-1} \zeta_{\Delta_{\mathbb{F}_q}(G_i)}(s).$$

In summary, we have the following

Proposition 3.13. *Suppose that $|G_i|$ is invertible in \mathbb{F}_q for $1 \leq i \leq d$. Then the zeta function of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is given by*

$$(1 - q^{-s})^{d-1} \prod_{i=1}^d \zeta_{\mathbb{F}_q[G_i]}(s).$$

We next consider the general case. We can assume that, up to an ordering, there exists a unique positive integer r such that

- $p \nmid |G_i|, 1 \leq i \leq r.$
- $p \mid |G_i|, r < i \leq d.$

We recall the following construction in [8, Section 5]. Let A be a general element of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$

$$A = \left[\begin{array}{c|c|c|c} C_1 & a_{12}J & \cdots & a_{1d}J \\ \hline a_{21}J & C_2 & \cdots & a_{2d}J \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d1}J & a_{d2}J & \cdots & C_d \end{array} \right].$$

We can further partition A into the following blocks

$$A = \begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix},$$

where A_1 is the union of the upper r blocks, A_2 is the union of the lower $d - r$ blocks, B_1 (respectively B_2) is the union of the upper right (respectively lower left) blocks. Concretely, we have

$$A_1 = \left[\begin{array}{c|c|c|c} C_1 & a_{12}J & \cdots & a_{1r}J \\ \hline a_{21}J & C_2 & \cdots & a_{2r}J \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{r1}J & a_{r2}J & \cdots & C_r \end{array} \right],$$

$$A_2 = \left[\begin{array}{c|c|c|c} C_{r+1} & a_{r+1,r+2}J & \cdots & a_{r+1,d}J \\ \hline a_{r+2,r+1}J & C_{r+2} & \cdots & a_{r+2,d}J \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,r+1}J & a_{d,r+2}J & \cdots & C_d \end{array} \right].$$

Similarly for B_1, B_2 . Note that we can consider A_1 (respectively A_2) as an element of $\mathcal{J}_{G_1, \dots, G_r}(\mathbb{F}_q)$ (respectively $\mathcal{J}_{G_{r+1}, \dots, G_d}(\mathbb{F}_q)$).

Theorem 3.14. ([8]) *Let I_i be the Jacobson radical of $\mathbb{F}_q[G_i]$. Let ψ be the map*

$$\psi: \mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q) \rightarrow \mathcal{J}_{G_1, \dots, G_r}(\mathbb{F}_q) \times \prod_{r+1 \leq i \leq d} \mathbb{F}_q[G_i]/I_i,$$

sending

$$A \mapsto (A_1, \overline{C_{r+1}}, \dots, \overline{C_d}).$$

Then ψ is a surjective ring homomorphism. Furthermore, the kernel of ψ is the Jacobson radical of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$. As a consequence,

$$\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)^{ss} \cong \mathcal{J}_{G_1, \dots, G_r}(\mathbb{F}_q) \times \prod_{r+1 \leq i \leq d} k[G_i]^{ss}.$$

Furthermore, by Corollary 2.12, we can further decompose

$$\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)^{ss} \cong M_r(\mathbb{F}_q) \times \prod_{i=1}^r \Delta_{\mathbb{F}_q}(G_i) \times \prod_{r+1 \leq i \leq d} k[G_i]^{ss}.$$

We have the following corollary by Theorem 3.14 and Proposition 3.13.

Theorem 3.15. *Let G_1, G_2, \dots, G_d be as above. Then the zeta function of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is given by*

$$\zeta_{\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)}(s) = (1 - q^{-s})^{r-1} \prod_{i=1}^d \zeta_{\mathcal{J}_{G_i}(\mathbb{F}_q)}(s).$$

4. q -rooted primes and the arithmetic of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$

In this section, we study the order of the unit group of the join algebra $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ where G_i is cyclic of order p_i with p_i a prime number different from q . This is a natural continuation of the work [6] where the author considers the case $d = 1$ and $q = 2$. We first recall the following definition.

Definition 4.1. Let p, q be two distinct prime numbers. We say that p is a q -rooted prime if q is a primitive root modulo p ; i.e., q is a generator of the multiplicative group \mathbb{F}_p^\times (which is a cyclic group of order $(p - 1)$). Equivalently, p is a q -rooted prime if and only if $\text{ord}_p(q) = p - 1$.

A conjecture of Emil Artin says that for any non-zero integer a other than $1, -1$ or a perfect square, there exist infinitely many primes p for which a is a primitive root mod p . In particular, this would imply that for a given prime q , there exist infinitely many p such that p is a q -rooted prime. This conjecture remains open, though some partial results are known. For example, R. Murty and R. Gupta proved unconditionally that there exists an integer that is a primitive root for infinitely many primes. D. R. Heath-Brown proved that at least one of $2, 3$, or 5 is a primitive root modulo infinitely many primes. Furthermore, it is known that Artin’s conjecture holds if we assume the Generalized Riemann Hypothesis. See [14,15] for further discussion on this topic.

In [6], the authors provide an elegant characterization of q -rooted primes using circulant matrices when $q = 2$. We remark, however, that their proof remains valid for any prime number q . For the sake of completeness, we provide the statement and complete proof here. In the subsequent discussion, the term “circulant matrices” specifically refers to G -circulant matrices where G is a cyclic group.

Theorem 4.2. *Let p be a prime number. Then, the following statements are equivalent.*

- (1) p is a q -rooted prime.
- (2) The order of the pole $s = 0$ of the zeta function $\zeta_{\mathbb{F}_q[\mathbb{Z}/p]}(s)$ is 2.
- (3) The order of the unit group of the group algebra $\mathbb{F}_q[\mathbb{Z}/p]$ is $(q^{p-1} - 1)(q - 1)$.
- (4) The number of invertible circulant matrices of size $p \times p$ over \mathbb{F}_q is $(q - 1)(q^{p-1} - 1)$.

Proof. The fact that (1) and (2) are equivalent follows directly from Corollary 3.9. For other parts, we observe that

$$\mathbb{F}_q[\mathbb{Z}/p] \cong \mathbb{F}_q[x]/(x^p - 1) = \mathbb{F}_q \times \mathbb{F}_q[x]/\Phi_p(x).$$

Here $\Phi_p(x) = \frac{x^p - 1}{x - 1}$ is the p -cyclotomic polynomial. By the proof of [6, Lemma 3.1], $\Phi_p(x)$ factors as a product of $m = \frac{p - 1}{\text{ord}_p(q)}$ distinct irreducible polynomials in $\mathbb{F}_q[x]$ of degree $n = \text{ord}_p(q)$. Consequently, as a ring, we have

$$\mathbb{F}_q[\mathbb{Z}/p] \cong \mathbb{F}_q \times \mathbb{F}_{q^m}.$$

We see that the order of the unit group of $\mathbb{F}_q[\mathbb{Z}/p]$ is given by $(q - 1)(q^n - 1)^m$. We also observe that

$$(q^n - 1)^m \leq q^{mn} - 1,$$

and the equality happens iff $m = 1$. This shows the equivalence of (1) and (3). The equivalence of (3) and (4) follows from the observation that units in the group ring correspond to invertible circulant matrices; see Proposition 2.2. \square

The following theorem is a direct generalization of Theorem 4.2.

Theorem 4.3. *Let q, p_1, p_2, \dots, p_d be prime numbers such that $p_i \neq q$. Then the following are equivalent*

- (1) p_i is a q -rooted prime for all $1 \leq i \leq d$.
- (2) The order of the pole $s = 0$ of the zeta function $\zeta_{\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)}$ is $d + 1$.
- (3) The order of the unit group of the join algebra $\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$ is

$$\prod_{i=1}^d (q^{p_i-1} - 1) \times \prod_{i=0}^{d-1} (q^d - q^i).$$

Proof. The equivalence between (1) and (2) follows from Theorem 4.2 and Proposition 3.13. Let us consider the other parts.

By Corollary 2.12 we know that the join algebra $\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$ is decomposed as

$$\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q) \cong M_d(\mathbb{F}_q) \times \prod_{i=1}^d \Delta_{\mathbb{F}_q}(\mathbb{Z}/p_i).$$

Consequently, the order of the unit group of the join algebra $\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$ is given by

$$|GL_d(\mathbb{F}_q)| \times \prod_{i=1}^d |\Delta_{\mathbb{F}_q}(\mathbb{Z}/p_i)^\times|.$$

By the proof of Theorem 4.2, we know that

$$|\Delta_{\mathbb{F}_q}(\mathbb{Z}/p_i)^\times| \leq q^{p_i-1} - 1,$$

with equality when p_i is a q -rooted prime. Combining this with the fact that

$$|GL_d(\mathbb{F}_q)| = \prod_{i=0}^{d-1} (q^d - q^i),$$

we get the equivalence of (1) and (3). This completes the proof of the theorem. \square

To motivate another characterization of q -rooted primes, consider the following question. *What are all units of order p in the ring $\mathbb{F}_q[\mathbb{Z}/p]$, where p and q are distinct primes?* The obvious units of order p that come to mind are of the form αg , where $\alpha \in \mathbb{F}_q$ and $g \in \mathbb{Z}/p$ such that $(\alpha g)^p = 1$ and $\alpha g \neq 1$. In the literature on group algebras, such units are called trivial units of order p . It turns out that these trivial units are the only possible units of order p precisely when q is p -rooted.

Proposition 4.4. *Let p and q be distinct primes. Every unit of order p in $\mathbb{F}_q[\mathbb{Z}/p]$ is trivial if and only if q is p -rooted.*

Proof. Recall the isomorphism:

$$\mathbb{F}_q[\mathbb{Z}/p] \cong \mathbb{F}_q \times \mathbb{F}_{q^n}^m,$$

where $m := \frac{p-1}{\text{ord}_p(q)}$ and $n := \text{ord}_p(q)$. Taking units on both sides, we get

$$(\mathbb{F}_q[\mathbb{Z}/p])^\times \cong \mathbb{Z}/(q-1) \times (\mathbb{Z}/(q^n-1))^m.$$

Every unit of order p in $\mathbb{F}_q[\mathbb{Z}/p]$ is trivial if and only if the number of trivial units of order p in $(\mathbb{F}_q[\mathbb{Z}/p])^\times$ is equal to the number of elements of order p in $\mathbb{Z}/(q-1) \times (\mathbb{Z}/(q^n-1))^m$.

Since the unit group in question is a finite abelian group, the number of elements of order p in $\mathbb{Z}/(q-1) \times (\mathbb{Z}/(q^n-1))^m$ is one less than the cardinality of the maximal elementary abelian p -subgroup in it.

Also note that, by definition of n , p divides $q^n - 1$. We consider two cases. If p does not divide $q - 1$, then equating the two numbers mentioned above, we get

$$p - 1 = p^m - 1.$$

Similarly, if p divides $q - 1$, we get

$$p^2 - 1 = p^{m+1} - 1.$$

In both cases, the equations are valid if and only if $m = 1$, or equivalently, q is p -rooted. \square

We now generalize this to the join of group rings. A unit u in $\mathcal{J}_{G_1, G_2, \dots, G_d}(k)$ is said to be a diagonal unit if $\epsilon(u)$ is a diagonal matrix; that means in our current situation that

all off-diagonal blocks of u must be zero. A diagonal unit is trivial if the i th diagonal block is of the form $\text{circ}([\alpha_i g_i])$, where $\alpha_i \in \mathbb{F}_q$ and $g_i \in G_i$ for $1 \leq i \leq d$. Note that when $d = 1$, this definition gives trivial units for group algebras. We are now ready to state the generalization.

Theorem 4.5. *Let q, p_1, p_2, \dots, p_d be prime numbers such that $p_i \neq q$. Every diagonal unit in $\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$ of order p is trivial if and only if p_i is a q -rooted prime for all $1 \leq i \leq d$.*

Proof. Observe that the subgroup of diagonal units in $\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$ is isomorphic to $\prod_{i=1}^d (\mathbb{F}_q[\mathbb{Z}/p_i])^\times$. From this, it follows that every diagonal unit in $\mathcal{J}_{\mathbb{Z}/p_1, \mathbb{Z}/p_2, \dots, \mathbb{Z}/p_d}(\mathbb{F}_q)$ of order p is trivial if and only if every unit of order p is trivial in $\mathbb{F}_q[\mathbb{Z}/p_i]$ for each i . Invoking the above proposition, we see that the latter holds if and only if p_i is a q -rooted prime for all $1 \leq i \leq d$. \square

5. $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ and Δ_{p^r} -rings

This section considers a special ring-theoretic property of the join ring $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$. Specifically, we are interested in the Δ_n -property of the join ring. To do so, we first recall the definition of a Δ_n -ring.

Definition 5.1. Let n be a positive integer. A ring R is said to be a Δ_n -ring if for each unit $u \in R^\times$, $u^n = 1$.

The Δ_n property of a ring is well-studied in the literature. It was first introduced in [4]. The author proves that the ring \mathbb{Z}/n of integers modulo n is a Δ_2 -ring if and only if n is a divisor of 24. In [7], the authors show that the ring $\mathbb{Z}/n[x_1, x_2, \dots, x_m]$ is a Δ_2 -ring if and only if n is a divisor of 12. Additionally, in [6], the authors classify all group algebras $k[G]$ which are a Δ_p -ring where G is an abelian group and p is a prime number (see [6, Theorem 1.4] and [6, Theorem 1.5].)

We remark that if R is a Δ_n -ring, it is also a Δ_m ring if $n|m$. If n is the smallest positive integer such that R satisfies this property, then call R a strict Δ_n -ring. We will frequently use the fact that whenever a Cartesian product of rings is a Δ_n -ring, so are all the individual factors of the product. We refer the reader to [4,6,7] for further discussions of this concept.

We next discuss the relationship between the Δ_n -property of R and its semisimplification R^{ss} . For this, we need the following proposition.

Proposition 5.2. *The canonical map $\Phi : R^\times \rightarrow (R^{ss})^\times$ is surjective.*

Proof. Let $a \in R^{ss}$ be a unit. Then there exists $b \in R^{ss}$ such that $ab = 1$. Let $a' \in R$ (respectively, $b' \in R$) be a preimage of a (respectively b). One has $a'b' = 1 + c$ for some $c \in \text{Rad}(R)$. Since $c \in \text{Rad}(R)$, $a'b' = 1 + c$ is right-invertible. This implies that a' is

right-invertible. Similarly, $b'a' = 1 + d$ for some $d \in \text{Rad}(R)$. From this, we deduce that a' is also left-invertible. Thus a' is a unit and $\Phi(a') = a$. This shows that Φ is surjective. \square

The following lemma follows directly from Proposition 5.2

Lemma 5.3. *If R is a Δ_n -ring then so is R^{ss} .*

We remark that the converse of Lemma 5.3 is not generally true. For example, let G be a 2-group such as $\mathbb{Z}/8$. Then $R = \mathbb{F}_2[G]$ is a local ring and the Jacobson radical of R is exactly the augmentation ideal $\Delta_{\mathbb{F}_2}(G)$ (see [3, Corollary 1.4]). Consequently $R^{ss} \cong \mathbb{F}_2$ which is a Δ_2 -ring. However, by [6, Theorem 1.4], we know that $\mathbb{F}_2[G]$ is not a Δ_2 -ring unless $G = (\mathbb{Z}/2)^r$.

When R is a field, we make the following observation that follows immediately from the fact that a polynomial of degree n over a field has at most n roots.

Lemma 5.4. *Let k be a field. If k is a Δ_n -ring, then k is a finite field.*

Remark 5.5. By Lemma 5.4, we can safely assume that all coefficient fields in the discussion below are finite.

This section will focus on Δ_{p^r} -rings where p is a prime number, and r is a positive integer. We remark that the case $r = 1$ was studied in [6], and our work here is a natural continuation of this line of research. We recall that for a group G , the exponent of G denoted by $\text{exp}(G)$ is the smallest integer n such that $g^n = 1$ for all $g \in G$. The following simple observation follows directly from the definition of a Δ_{p^r} -ring.

Proposition 5.6. *If R is a Δ_{p^r} -ring then R^\times is a p -group with exponent at most p^r .*

Here is an observation that we will use throughout this section.

Lemma 5.7. *Let q, p^r be two prime powers. The matrix algebra $M_n(\mathbb{F}_q)$ is a Δ_{p^r} -ring if and only if $n = 1$ and \mathbb{F}_q is a Δ_{p^r} -ring.*

Proof. Let us assume that $M_n(\mathbb{F}_q)$ is a Δ_{p^r} -ring. By Proposition 5.6, we know that $GL_n(\mathbb{F}_q)$ must be a p -group. Additionally, we know that the order of $GL_n(\mathbb{F}_q)$ is

$$\prod_{i=0}^{n-1} (q^n - q^i) = \prod_{i=0}^{n-1} q^i (q^{n-i} - 1).$$

Suppose that $n \geq 2$. We see that $|GL_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1} (q^n - q^i)$ has at least two distinct prime factors. This shows that $GL_n(\mathbb{F}_q)$ is not a p -group which contradicts the fact that $M_n(\mathbb{F}_q)$ is a Δ_{p^r} -ring. \square

The main goal of this section is to classify all join algebras $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ which are Δ_{p^r} -rings. To begin this study, we start with the simplest case, namely $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is \mathbb{F}_q (this corresponds to the case $d = 1$ and $G_1 = \{e\}$ the trivial group). To answer this question, we first recall the famous Catalan conjecture, now a theorem of Mihailescu (see [22,30].)

Theorem 5.8. (See [22]) *The only solution in the natural numbers of the Diophantine equation*

$$x^a - y^b = 1,$$

where $a, b > 1$ and $x, y > 0$ is $x = 3, a = 2, y = 2, b = 3$.

Here is a direct corollary of this theorem, which is a generalization of [6, Lemma 2.1] (see [5, Theorem 2.4] for a different but equivalent statement.)

Corollary 5.9. *Let q be a prime power. Then the finite field \mathbb{F}_q is a Δ_{p^r} -ring if and only if one of the following conditions hold:*

- (1) $p = 2, q = 2^{2^n} + 1$ is a Fermat prime, and $r \geq 2^n$. In this case, \mathbb{F}_q is a strict $\Delta_{2^{2^n}}$ -ring.
- (2) $p = 2^a - 1$ is a Mersenne prime and $q = p + 1 = 2^a$. In this case \mathbb{F}_q is a strict Δ_p -ring.
- (3) $p = 2, q = 9$, and $r \geq 3$. In this case, \mathbb{F}_q is a strict Δ_8 -ring.
- (4) $q = 2, p$ and r are arbitrary.

Proof. The unit group \mathbb{F}_q^\times of \mathbb{F}_q is a cyclic group of order $q - 1$. Consequently, \mathbb{F}_q is a Δ_{p^r} -ring if and only if $q - 1 | p^r$. Since p is a prime number, there exists $0 \leq b \leq r$ such that $q - 1 = p^b$. Let us write $q = x^a$ where x is a prime and a is a positive integer. We then have the following Diophantine equation

$$x^a - p^b = 1.$$

If $a, b > 1$, then by Mihailescu’s Theorem 5.8, we know that $x = 3, a = 2, p = 2, b = 3$, thereby satisfying condition (3). If $b = 0$ then $x = 2, a = 1$. Consequently, $q = 2$, satisfying condition (4). Next, we consider the case $a = 1$. Then $x = p^b + 1$. Since $x > 2$, it must be odd. As a result, p is even, hence $p = 2$. Therefore, $x = 2^b + 1$. From here, we can deduce that $b = 2^n$ and $x = 2^{2^n} + 1$ is a Fermat prime. Thus, in this case, condition (1) is satisfied. Finally, let us consider the case $b = 1$. Then we have $p = x^a - 1$. If $p = 2$ then $x = 3, a = 1$, and we again satisfy condition (1). So, we can safely assume that p is odd. As a result, $x = 2$ and $p = 2^a - 1$ is a Mersenne prime, which fulfills condition (2). \square

Next, we will answer the following question: For which groups G is the group algebra $\mathbb{F}_q[G]$ a Δ_{p^r} -ring? From the canonical embedding $G \hookrightarrow \mathbb{F}_q[G]^\times$, we conclude that if $\mathbb{F}_q[G]$ is an Δ_{p^r} -ring then G must be a p -group. It turns out that in most cases, G must also be abelian. More precisely, we have the following proposition.

Proposition 5.10. *Assume that $(p, q) \neq (2, 2)$ and that $\mathbb{F}_q[G]$ is a Δ_{p^r} -ring. Then G is an abelian p -group.*

Proof. Since $\mathbb{F}_q \subset \mathbb{F}_q[G]$, we conclude that \mathbb{F}_q is also a Δ_{p^r} -ring. Since $(p, q) \neq (2, 2)$, Corollary 5.9 implies that $\gcd(p, q) = 1$. Since G is a p -group, $|G|$ is invertible in \mathbb{F}_q . By Maschke’s theorem, $\mathbb{F}_q[G]$ is semisimple and by the Artin-Wedderburn theorem we must have

$$\mathbb{F}_q[G] \cong \prod_{i=1}^r M_{n_i}(D_i),$$

where D_i is a division algebra over \mathbb{F}_q . Since \mathbb{F}_q is a finite field, D_i is a finite field as well; see [11, Chapter 13, Exercise 13, Page 536]. By Lemma 5.7, we conclude that $n_i = 1$ and D_i is an Δ_{p^r} -algebra for all $1 \leq i \leq r$. This implies that $\mathbb{F}_q[G]$ is abelian and hence G is also abelian. \square

We now deal with the case $(p, q) = (2, 2)$ separately. Here, instead of working with this particular case, we discuss a more general study of modular group rings, which might be of independent interest. Let k be a finite field of characteristic p and G a finite p -group. Let $\Delta_k(G)$ be the augmentation ideal. It is known that $\Delta_k(G)$ is a nilpotent ideal; in fact $\Delta_k(G)^{|G|} = 0$ (see [3, Corollary 1.3]). Let $U_1(k[G]) := 1 + \Delta_k(G)$ be the set of all normalized units in $k[G]$. We remark that if $u = 1 + x \in U_1(k[G])$ with $x \in \Delta_k(G)$ then

$$u^{|G|} = 1 + x^{|G|} = 1.$$

This shows that $U_1(k[G])$ is a p -group. From the isomorphism $k[G]^\times \cong k^\times \times U_1(k[G])$, we conclude that $k[G]^\times$ is a p -group if and only if k^\times is a p -group. Since $\text{char}(k) = p$, this happens if and only if $k = \mathbb{F}_2$. In summary, we have

Proposition 5.11. *Let $(p, q) = (2, 2)$ and G a 2-group. Then $\mathbb{F}_q[G]$ is a Δ_{2^r} -ring where $2^r = \exp(U_1(\mathbb{F}_q[G]))$. Furthermore, if G is abelian, $\mathbb{F}_q[G]$ is a strict $\Delta_{\exp(G)}$ -ring.*

Proof. We already explained the proof of the first part. For the second part, we note that if G is abelian and

$$u = 1 + x = \sum_{g \in G} a_g g$$

is a normalized unit (so $a_e = 1$), then

$$u^{\exp(G)} = \sum_{g \in G} a_g^{\exp(G)} g^{\exp(G)} = \sum_{g \in G} a_g = 1. \quad \square$$

Remark 5.12. It is worth mentioning that the problem of determining $\exp(U_1(\mathbb{F}_2[G]))$ is well-studied but still open in the literature. For further discussions, see [17,31].

With these preliminary results, we are now ready to classify all group algebras $\mathbb{F}_q[G]$ which are Δ_{p^r} -rings.

Theorem 5.13. *Let q, p^r be prime powers. Let G be a finite group. The group algebra $\mathbb{F}_q[G]$ is a Δ_{p^r} -ring if and only if G is a p -group and one of the following conditions holds.*

- (1) $p = 2, q = 2^{2^n} + 1$ is a Fermat prime, $r \geq 2^n$, G is abelian, and the exponent of G is a divisor of 2^{2^n} . In this case, $\mathbb{F}_q[G]$ is a strict $\Delta_{2^{2^n}}$ -ring.
- (2) $p = 2^a - 1$ is a Mersenne prime, $q = p + 1 = 2^a$ and $G = (\mathbb{Z}/p)^s$ for some $s \geq 0$. In this case $\mathbb{F}_q[G]$ is a strict Δ_p -ring.
- (3) $p = 2^a - 1$ is a Mersenne prime, $q = 2$ and $G = (\mathbb{Z}/p)^s$ for some $s \geq 0$.
- (4) $p = 2, q = 3, r \geq 3, G$ is abelian, and the exponent of G is 4 or 8.
- (5) $p = 2, q = 9, r \geq 3, G$ is abelian, and the exponent of G is at most 8. In this case, $\mathbb{F}_q[G]$ is a strict Δ_8 -ring.
- (6) $q = 2, p = 2$ and $2^r \geq \exp(U_1(\mathbb{F}_2[G]))$.

Proof. We will discuss both directions of the above theorem simultaneously. First, note that since \mathbb{F}_q is a subring of $\mathbb{F}_q[G]$, if $\mathbb{F}_q[G]$ is a Δ_{p^r} -ring then so is \mathbb{F}_q . On the other hand, from Corollary 5.9, $\mathbb{F}_q[G]$ is a Δ_{p^r} -ring if any of conditions (1)-(4) hold. Thus, we may assume that $\mathbb{F}_q[G]$ is a Δ_{p^r} -ring (since otherwise, neither side of our equivalence holds). Also from Corollary 5.9, $p \neq \text{char}(\mathbb{F}_q)$ unless $p = q = 2$. The case $(p, q) = (2, 2)$ is treated separately in Proposition 5.11. For now, let us assume that $(p, q) \neq (2, 2)$. By Proposition 5.10, we conclude that G is abelian. Since G is an abelian p -group with $\text{gcd}(p, q) = 1$, Theorem 3.8 implies

$$\mathbb{F}_q[G] \cong \bigoplus_{d|G} a_d \mathbb{F}_q[\zeta_d].$$

Here ζ_d is a primitive root of unity of order d and $a_d = \frac{n_d}{[\mathbb{F}_q(\zeta_d) : \mathbb{F}_q]}$ where n_d is the number of elements of order d in G . From this formula, we conclude that $\mathbb{F}_q[G]$ is a Δ_{p^r} -ring if and only if each component $\mathbb{F}_q[\zeta_d]$ is. Since $|G|$ is a p -group and $\mathbb{F}_q[\zeta_{d'}] \subset \mathbb{F}_q[\zeta_d]$ if $d' | d$, we conclude that $\mathbb{F}_q[G]$ is a Δ_{p^r} -ring if and only if $\mathbb{F}_q[\zeta_D]$ is Δ_{p^r} -ring where D is largest number such that $n_D > 0$. Since G is a p -group, D is exactly the exponent of G . We remark that $\mathbb{F}_q[\zeta_D] = \mathbb{F}_{q^m}$ where

$$m = [\mathbb{F}_q(\zeta_D) : \mathbb{F}_q] = \text{ord}_D(q).$$

We now consider a few cases based on the classification described in Corollary 5.9.

Case 1: $p = 2$ and $q^m = 2^{2^n} + 1$ is a Fermat prime. This shows that $m = 1$ and q is a Fermat prime. Furthermore, $m = 1$ means that $\text{ord}_D(q) = 1$ or equivalently $D|q - 1 = 2^{2^n}$. This covers the first case of our theorem.

Case 2: $p = 2^a - 1$ is a Mersenne prime and $q^m = p + 1 = 2^a$. This shows that $q = 2^b$ with $bm = a$. By definition of m , we have $q^m \equiv 1 \pmod{D}$. Since $q^m = 2^a$, this is equivalent to $D|2^a - 1 = p$. This implies $D = 1$ or $D = p$. From this, we can conclude that $G = (\mathbb{Z}/p)^s$ for some $s \geq 0$. Furthermore, we remark that since $p = 2^a - 1$ is a prime number, a is a prime number. We then see that $(b, m) = (a, 1)$ or $(b, m) = (1, a)$. The case $(b, m) = (a, 1)$ covers the second case of our theorem and the case $(b, m) = (1, a)$ covers the third case of our theorem.

Case 3: $p = 2$, $q^m = 9$ and $r \geq 3$. First, consider the case where $(q, m) = (3, 2)$. Since $m = 2$, we know that $9 = q^m \equiv 1 \pmod{D}$ and $q \not\equiv 1 \pmod{D}$. This shows that $D \in \{4, 8\}$. This covers the fourth case of our theorem. Next, consider the case $(q, m) = (9, 1)$. Again, we see that $D|8$. This covers the fifth case of our theorem. \square

We now focus on a special case.

Definition 5.14. A ring R is said to have the *diagonal property* if it is a Δ_2 -ring.

The classification given in Theorem 5.13 provides another proof for the following statements, which were first proved in [6] under the assumption that G is abelian.

Corollary 5.15. ([6, Theorem 1.4] and [6, Theorem 1.5]) *Let G be a group and k a field.*

- (1) *The group algebra $k[G]$ has the diagonal property if and only if $k[G]$ is either $\mathbb{F}_2[(\mathbb{Z}/2)^r]$ or $\mathbb{F}_3[(\mathbb{Z}/2)^r]$.*
- (2) *Let p be an odd prime. The group algebra $k[G]$ is a Δ_p -ring if and only if p is a Mersenne prime and $k[G]$ is either $\mathbb{F}_2[(\mathbb{Z}/p)^r]$ or $\mathbb{F}_{p+1}[(\mathbb{Z}/p)^r]$.*

Thus, these results give us a simple and elegant characterization of Mersenne primes. An odd prime p is a Mersenne if and only if $\mathbb{F}_2[E]$ is a Δ_p -ring, where E is any finite elementary abelian p -group.

Finally, we answer the following question: which join algebra $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is a Δ_{p^r} -ring.

Theorem 5.16. *Suppose that $d \geq 2$. Then the join algebra $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is a Δ_{p^r} -ring if and only if the following conditions are satisfied*

- (1) $p = q = 2$.
- (2) G_i is a 2-group for all $1 \leq i \leq d$.
- (3) There is at most one index i such that $G_i = \{e\}$ the trivial group.
- (4) $2^r \geq \max_{1 \leq i \leq d} \exp(U_1(\mathbb{F}_2[G_i]))$.

Proof. Let us prove the “only if” part of the above theorem. So assume $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is a Δ_{p^r} -ring. First, we claim that $(p, q) = 2$. In fact, suppose that $(p, q) \neq (2, 2)$. Let us consider the following embedding $\mathbb{F}_q[G_d]^\times \hookrightarrow \mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)^\times$ sending

$$C_1 \mapsto \left[\begin{array}{c|c|c|c} C_1 & 0 & \cdots & 0 \\ \hline 0 & I_{k_2} & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & I_{k_d} \end{array} \right].$$

Since $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is a Δ_{p^r} -ring, $k[G_1]$ is a Δ_{p^r} -ring as well. Similarly, $k[G_i]$ is a Δ_{p^r} -ring for all $1 \leq i \leq d$. We conclude that G_i is a p -group for all $1 \leq i \leq d$. Furthermore, by Theorem 5.13, we know that $\gcd(p, q) = 1$ since we assume that $(p, q) \neq (2, 2)$. Then by Corollary 2.12, $M_d(\mathbb{F}_q)$ is a direct factor of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$. This shows that $M_d(\mathbb{F}_q)$ is a Δ_{p^r} -ring as well. However, Lemma 5.7 implies that $d = 1$, which is a contradiction. This shows that $(p, q) = (2, 2)$.

From now on, we will assume that $(p, q) = 2$. In particular, this implies that G_i is a 2-group. Suppose there are exactly t elements amongst G_i , which are trivial groups. We claim that $t \leq 1$. In fact, by Theorem 3.14, $M_t(\mathbb{F}_2)$ is a direct factor of $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_2)^{\text{ss}}$ which is a Δ_{p^r} -ring by Lemma 5.3. This shows that $M_t(\mathbb{F}_2)$ is a Δ_{p^r} -ring. By Lemma 5.7, we conclude that $0 \leq t \leq 1$. Finally, the embedding $\mathbb{F}_2[G_i]^\times \hookrightarrow \mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_2)^\times$ explained above implies that

$$2^r \geq \exp(U_1(\mathbb{F}_2[G_i])), \forall 1 \leq i \leq d.$$

In summary, we have proved the “only if” part of the theorem. We now prove the converse. Let us consider the case that all G_i are nontrivial 2-groups. Let

$$A = \left[\begin{array}{c|c|c|c} C_1 & a_{12}J & \cdots & a_{1d}J \\ \hline a_{21}J & C_2 & \cdots & a_{2d}J \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d1}J & a_{d2}J & \cdots & C_d \end{array} \right]$$

be an invertible element in $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_2)$. Then $\epsilon(A)$ is invertible where $\epsilon : \mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q) \rightarrow M_d(\mathbb{F}_2)$ is the augmentation map. By definition, we have

$$\epsilon(A) = \left[\begin{array}{c|c|c|c} \epsilon(C_1) & 0 & \cdots & 0 \\ \hline 0 & \epsilon(C_2) & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \epsilon(C_d) \end{array} \right] \in GL_d(\mathbb{F}_2).$$

We conclude that $\epsilon(C_i) = 1$ for all $1 \leq i \leq d$. This implies that C_i is invertible for $1 \leq i \leq d$ since $\mathbb{F}_2[G_i]$ is a local ring in which $\Delta_{\mathbb{F}_2}(G_i)$ is the maximal ideal. We then see that

$$A^2 = \left[\begin{array}{c|c|c|c} C_1^2 & 0 & \cdots & 0 \\ \hline 0 & C_2^2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & C_d^2 \end{array} \right].$$

Consequently

$$A^{2^r} = \left[\begin{array}{c|c|c|c} C_1^{2^r} & 0 & \cdots & 0 \\ \hline 0 & C_2^{2^r} & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & C_d^{2^r} \end{array} \right].$$

Since $\mathbb{F}_2[G_i]$ is a Δ_{2^r} -ring, we conclude that $C_i^{2^r} = I$. As a result, $A^{2^r} = I$. This shows that $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_2)$ is a Δ_{2^r} -ring. The case where there is one $G_i = \{e\}$ can be proved using similar calculations. \square

A direct corollary of the above theorem is the following.

Corollary 5.17. *Suppose that $d \geq 2$. Then the join algebra $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ has the diagonal property if and only if $\mathcal{J}_{G_1, G_2, \dots, G_d}(\mathbb{F}_q)$ is $\mathcal{J}_{(\mathbb{Z}/2)^{r_1}, (\mathbb{Z}/2)^{r_2}, \dots, (\mathbb{Z}/2)^{r_d}}(\mathbb{F}_2)$ where $r_i \in \mathbb{Z}_{\geq 0}$ and at most one of the r_i is equal to 0.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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