ON U-UNITARY CAYLEY GRAPHS OVER FINITE RINGS

TUNG T. NGUYEN, NGUYỄN DUY TÂN

ABSTRACT. Graphs defined over a finite ring are well-studied in the literature. Due to their nature, these types of graphs connect several branches of mathematics, including algebra, number theory, matrix theory, and representation theory. In a recent work, we study U-unitary Cayley graphs over a finite commutative ring, which unifies several constructions of graphs with arithmetic origins. Among various structural graph-theoretic results on these graphs—such as their connectedness, primeness, and bipartiteness—we show that their spectra can be described via a certain supercharacter theory. Utilizing this spectral description, we are able to find some classes of gcd-graphs that possess perfect state transfer. In this article, we generalize this study to finite non-commutative rings, with a special focus on the case of the matrix rings with coefficients in a finite field. We show, in particular, that gcd-graphs over these matrix rings have no perfect state transfer.

1. INTRODUCTION

Let n be a positive integer and D a subset of proper divisors of n. The gcd-graph $G_n(D)$ is defined as follows:

(1) The vertices of $G_n(D)$ are elements of the finite ring $\mathbb{Z}/n$.
(2) Two vertices a, b are adjacent if $\gcd(a-b, n) \in D$.

This type of graph was first introduced by Klotz and Sander in [18]. There, the authors describe several fundamental graph-theoretic properties of these graphs. In particular, they explain a beautiful connection between the spectra of these graphs and Ramanujan sums. As a consequence of this spectral description, Klotz and Sander show that all eigenvalues of gcd-graphs are integers. In [33], So proves the converse of this statement; namely, if a Cayley graph over $\mathbb{Z}/n$ has all integral eigenvalues, then it must be a gcd-graph. The works of Klotz, Sander, and So have led to a series of studies on Perfect State Transfer (PST) on graphs—a concept introduced by physicists in quantum spin networks (see [2, 3, 32, 35]).

In general, a gcd-graph can be defined over a finite commutative ring R using the interplay between the additive and multiplicative structures of R (see [20, 24, 25, 35]).

2020 *Mathematics Subject Classification.* Primary 05C25, 11L05, 13A70, 13M05.

Key words and phrases. Gcd graphs, Finite rings, Supercharacter theory, Perfect State Transfer.

TTN is partially supported by an AMS-Simons Travel Grant. NDT is partially supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2023.21.

Many aspects of these gcd-graphs have been investigated, including but not limited to their connectedness, primeness, and clique and independence numbers (see [1, 35, 24, 26]). Furthermore, when the underlying ring R is a finite commutative Frobenius ring, we show in a previous work that the spectra of gcd-graphs over R can be described by various arithmetical sums such as Gauss sums, Ramanujan sums, and Heilbronn sums (see [28]).

The goal of this article is to define and study the concept of gcd-graphs over arbitrary finite rings (and, more generally, U-unitary Cayley graphs over these rings). While some former results over commutative rings generalize straightforwardly, others require more careful consideration, as left and right multiplication might not be the same that leads to several new phenomenon such as two-sided ideal structures. In particular, this necessitates a revisit of the notion of a gcd-graph in a non-commutative setting.

1.1. **Outline.** In Section 2, we introduce the notion of a U-unitary Cayley graph over a finite ring R. We then describe some foundational graph-theoretic properties of these graphs, including their connectedness and primeness. In particular, we provide a complete answer regarding the connectedness and primeness of the unitary Cayley graph associated with R under some mild conditions. Section 3 focuses on the spectra of these U-unitary Cayley graphs when the underlying R is a symmetric Frobenius ring. We show that these spectra can be described via certain supercharacter theory on R. As a consequence of this spectral description, we provide an upper bound for the number of distinct eigenvalues in a U-unitary Cayley graph. For certain graphs over a matrix ring $M_n(F)$, we also calculate this upper bound explicitly. In Section 3, we also study the notation of relative Frobenius rings, which might be of independent interest. Finally, we utilize this spectral description to study the existence of Perfect State Transfer (PST) on U-unitary Cayley graphs. We show, in particular, that PST cannot exist on any gcd-graph over $M_n(F)$.

2. U-UNITARY CAYLEY GRAPHS AND THEIR GRAPH THEORETIC PROPERTIES

2.1. **U-unitary Cayley graphs.** Let R be a finite ring and S a subset of $(R, +)$ such that $S = -S$ and S does not contain 0. The Cayley graph $\Gamma(R, S)$ is the undirected simple graph whose vertex set is R, and two vertices a and b are adjacent if and only if $a - b \in S$. In practice, S is often referred to as the connecting set of $\Gamma(R, S)$. Furthermore, in many applications, S often has an arithmetic origin (see [1, 14, 15, 17, 22, 31, 30, 34] for some works in this line of research).

When R is commutative, we define in [28] the notion of a U-unitary Cayley graph, where U is a subgroup of $R^\times$—the unit group of R. More precisely, a Cayley graph of the form $\Gamma(R, S)$ is called a U-unitary Cayley graph if S is stable under the action of U; namely, $US = S$. As explained in [28], when $U = R^\times$ and $R = \mathbb{Z}/n$, this definition coincides with the classical definition of a gcd-graph described in the introduction. We remark that, by its definition, U acts as an automorphism of each U-unitary Cayley graph $\Gamma(R,S)$ (see [28, Proposition 3.7]).

When R is not commutative, we must take into account the fact that left and right multiplication might not be the same. Consequently, it seems reasonable to consider them simultaneously. For this reason, we introduce the following definition.

Definition 2.1. Let R, S be as before. Let U be a subgroup of $R^\times$ such that $-1 \in U$. We say that $\Gamma(R,S)$ is a U-unitary Cayley graph if S is stable under the left and right action of U; namely, $USU = S$. Note that since $-1 \in U$, this condition automatically implies that S is symmetric.

Remark 2.2. As explained in [23, Proposition 2.1], an element $r \in R$ is left or right invertible if and only if it is invertible. Therefore, there is no left/right ambiguity in the definition of U and $R^\times$.

Remark 2.3. When $U = R^\times$, we will use the term $R^\times$-unitary Cayley graph and gcd-graph interchangeably.

We define the following relation on R: we say that $x \sim_U y$ if $x = u_1 y u_2$ where $u_1, u_2 \in U$. We can see that this is an equivalence relation. Furthermore, if we denote by I_x (respectively I_y) the two-sided ideal generated by x (respectively y) then $I_x = I_y$. Here, we recall that the two-sided ideal I_x is the set of elements of the forms $\sum_{i=1}^{n} a_i x b_i$ where $a_i, b_i \in R$.

Let $\mathcal{K} = \{K_1, K_2, \ldots, K_m\}$ be the elements of the double quotient $U \backslash R / U$. In other words, $\mathcal{K}$ is precisely the equivalent classes of R with respect to $\sim_U$. By the definition, we have the following criterion for a graph to be U-unitary.

Proposition 2.4. *The following conditions are equivalent.*

(1) $\Gamma(R,S)$ is a U-unitary graph.
(2) S is a disjoint union of some orbits K_i.

Corollary 2.5. *If $\Gamma(R,S)$ is a U-unitary Cayley graph, then its complement graph $\Gamma(R,S)^c$ is also an unitary Cayley graph.*

Proof. By definition, $\Gamma(R,S)^c$ is precisely $\Gamma(R,S')$ where $S' = R \setminus (S \cup \{0\}) = \mathcal{K} \setminus (S \cup \{0\})$. Since S and $\{0\}$ are both unions of some orbits K_i's, the same holds true for S'. □

2.2. Graph-theoretic properties of U-unitary Cayley graphs.

2.2.1. *Connectedness.* In this section, we answer the following question: when is a U-unitary Cayley graph connected? To do so, we first introduce the following convention: for each $1 \leq i \leq m$, let I_i be the ideal generated by an element of $x \in K_i$ (by definition of K_i, I_i is independent of the choice of x). Let $\ell(K_i)$ be the smallest numbers such that every element y in I_i be written in the form $y = \sum_{i=1}^{\ell(K_i)} a_i x b_i$ where $a_i, b_i \in R$. We note that since R is finite, such an a number always exists. Furthermore, if R is commutative, then $\ell(K_i)$ is either 0 or 1. We now state the main result for the connectedness of a U-unitary Cayley graph.

Proposition 2.6. *Let $\Gamma(R,S)$ be a U-unitary Cayley graph. Suppose that $\Gamma(R,U)$ is connected. Then the following conditions are equivalent.*

(1) $\Gamma(R,S)$ is connected.

(2) $R=\sum_i I_i$ where the sum is over all ideals I_i such that $K_i \subset S$.

Furthermore, if one of these conditions are satisfied, then

$$diam(\Gamma(R,S)) \leq diam(\Gamma(R,U))^2 \sum_{s=1}^{t} \ell(K_{i_s}).$$

Here t is the smallest positive integer in which there exists $i_1, i_2, \ldots, i_t$ such that $R=\sum_{s=1}^{t} I_{i_s}$ and $S \cap K_{i_s} \neq \emptyset$.

Proof. The proof for this statement is a slight modification of the one given for the commutative case in [28, Proposition 3.10]. We remark, however, that in this case the upper bound for the diameter of $\Gamma(R,S)$ is slightly larger due to the the non-commutativity of R.

We first show that (1) $\implies$ (2). Indeed, because $\Gamma(R,S)$ is connected, we can find a $\mathbb{Z}$-linear combination: $1=\sum_i a_i s_i$, where $a_i \in \mathbb{Z}$ and $s_i \in S$. By definition of I_i, we know that $\sum_i a_i s_i \in \sum_i I_i$. This shows that $1 \in \sum_i I_i$, which implies that $R=\sum_i I_i$.

Conversely, suppose that (2) holds. We will show that $\Gamma(R,S)$ is connected. In fact, let $r \in R$. Since $R=\sum_{s=1}^{t} I_{i_s}$, we can write

$$r=\sum_{s=1}^{t}\left(\sum_{j=1}^{\ell(K_{i_s})} a_{i_s j} x_{i_s} b_{i_s j}\right), \tag{2.1}$$

where $x_{i_s} \in K_{i_s}$, $n_s \in \mathbb{N}$, and $a_{i_s j}, b_{i_s j} \in R$. Furthermore, since $\Gamma(R,U)$ is connected, each $a_{i_s j}$ and $b_{i_s j}$ can be written as a sum of at most $\mathrm{diam}(\Gamma(R,U))$ elements in U. Consequently, each term $a_{i_s j} x_{i_s} b_{i_s j}$ can be written as a $\mathbb{Z}$-linear combinations of at most $\mathrm{diam}(\Gamma(R,U))^2$ elements in K_{i_s}. From Eq. (2.1), we conclude that

$$\mathrm{diam}(\Gamma(R,S)) \leq \mathrm{diam}(\Gamma(R,U))^2 \sum_{s=1}^{t} \ell(K_{i_s}).$$

□

Corollary 2.7. *Let $R=M_n(F)$ where $n>1$ and F is a finite field. Let $U \subset R^\times = GL_n(F)$ such that $\Gamma(R,U)$ is connected. Let $\Gamma(R,S)$ be a U-unitary Cayley graph such that S is not an empty set. Then $\Gamma(R,S)$ is connected.*

Proof. This follows from Proposition 2.6 and the fact that the only two-sided ideals of $M_n(F)$ are 0 and $M_n(F)$. □

Remark 2.8. By [19], every square matrix of size $n \times n$ with $n > 1$ is the sum of two invertible matrices. Therefore, the graph $\Gamma(M_n(F), GL_n(F))$ is always connected. The above corollary shows that all non-empty $GL_n(F)$-unitary Cayley graphs are connected.

It turns out that every non-empty $SL_n(F)$-graph is connected.

Proposition 2.9. *The Cayley graph* $\Gamma(M_n(F), SL_n(F))$ *is connected.*

Remark 2.10. We remark that since we only work with undirected graph, we assume implicitly here that $-I_n \in SL_n(F)$. This happens only if $\text{char}(F) = 2$ or n is even.

Proof. By the proof of Proposition 3.10, for two singular matrices A, B, $A \sim_{GL_n(F)} B$ if and only if $A \sim_{SL_n(F)} B$. Furthermore, since every matrix is a sum of matrices with exactly one non-zero element, it is sufficient to show that for each $a \in F$, the following can be written as a sum of elements in $SL_n(F)$

$$A = a \oplus 0_{n-1} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

In fact, let X and Y be as follow

$$X = \begin{bmatrix} 1+a & 1 \\ -1 & 0 \end{bmatrix}, X' = X \oplus I_{n-2},$$

$$Y = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, X' = Y' \oplus I_{n-2},$$

Then $A = X' + (-Y')$ and $\det(X') = \det(-Y') = 1$. We remark that we implicitly use the assumption that $\det(-I_n) = 1$. ☐

2.2.2. *Primeness.* In this section, we study the primeness of a U-unitary Cayley graphs. First, we need to recall this definition.

A subset X in a graph G is called a homogeneous set if every vertex in $V(G) \setminus X$ is adjacent to either all or none of the vertices in X. By definition, if $X = V(G)$ or $|X| = 1$ then X is a homogeneous set– it is called a trivial homogeneous set. Otherwise, a homogeneous set X with $2 \leq X < |V(G)|$ is called non-trivial. The graph G is said to be prime if it does not contain any non-trivial homogeneous sets. We note that, by definition, a set X is homogeneous in G if and only if X is homogeneous in the complement G^c of G. Additionally, we also note that the notion of a homogeneous set generalizes the notion of a connected component; namely, a connected component of a graph is always a homogeneous set. For this reason, when studying prime graphs, we can safely assume that $\Gamma(R, S)$ and its complement are both connected.

As explained in previous works such as [9, 25, 28], the existence of a a homogeneous set on a Cayley graph requires some rather strong conditions on the generating set. In particular, in [28], we show that if a U-unitary Cayley graph over a commutative ring is not prime, then there exists a proper ideal I ; namely $I \neq 0$ and $I \neq R$, such that I is a homogeneous set. This statement can be generalized to the non-commutative setting.

Proposition 2.11. *Suppose that $\Gamma(R, U)$ is connected. Suppose further that $\Gamma(R, S)$ is both connected and anti-connected. Then, the following conditions are equivalent.*

(1) $\Gamma(R, S)$ is not a prime graph.

(2) There exists a proper two-sided ideal I in R such that I is a homogeneous set in $\Gamma(R, S)$.

Proof. By definition, $(2) \implies (1)$. Let us show that $(1) \implies (2)$. By [9, Theorem 3.4], if I is a maximal non-trivial homogeneous set of $\Gamma(R, S)$ containing 0, then I is a subgroup of $(R, +)$. We claim that I is a left ideal in R as well. For each $u \in U$, the left multiplication by u is an automorphism of $\Gamma(R, S)$. Consequently, uI is also a homogeneous set. Since $0 \in I \cap uI$ $I \cup uI$ is also a homogeneous set (see [9, Lemma 3.1]). By the maximality of I, we must have $uI = I$. We conclude that I is stable under the left action of U. We now show that if $r \in R$, then $rI \subset I$. In fact, since $\Gamma(R, U)$ is connected, we can write $r = \sum_{i=1}^{d} m_i u_i$, where $m_i \in \mathbb{Z}$ and $u_i \in U$. For each $h \in I$, we have $rh = \sum_{i=1}^{d} m_i(u_i h)$. Since $u_i h \in I$ and I is a subgroup of $(R, +)$, we conclude that $rh \in I$. This shows that $rI \subset I$ for all $r \in R$. Therefore, I is a left ideal in R. An identical argument show that I is also a right ideal in R. We conclude that I is a two-sided ideal in R. □

Let us discuss some corollaries of Proposition 2.11.

Corollary 2.12. *Let $R = M_n(F)$ with $n > 2$ and F is a finite field. Let $U \subset R^\times = GL_n(F)$ such that $\Gamma(R, U)$ is connected. Let $\Gamma(R, S)$ be a U-unitary Cayley graph such that $S \neq \emptyset$ and $S \neq M_n(F) \setminus \{0\}$ (equivalently $\Gamma(R, S)$ is not a complete or empty graph). Then $\Gamma(R, S)$ is prime. In particular, if $U = GL_n(F)$ or $U = SL_n(F)$, then $\Gamma(R, S)$ is always prime.*

Proof. By Corollary 2.7, we know that $\Gamma(R, S)$ is both connected an anti connected. Furthermore, $M_n(F)$ has no proper two-sided ideals. Therefore, Proposition 2.11 shows that $\Gamma(R, S)$ is prime. □

In the case $S = U$, we have the a rather strict condition when $\Gamma(R, U)$ is not prime. To discuss this restriction, we recall that the Jacobson radical $\mathrm{Rad}(R)$ of R is the intersection of all left maximal ideals in R (see [29, Chapter 4.3]). It is known that $\mathrm{Rad}(R)$ is a two-sided ideal in R.

Proposition 2.13. *Let I be a left (or right) ideal which is also a homogeneous set in $\Gamma(R, U)$. Then $I + U \subset U$. Furthermore, $I \subset \mathrm{Rad}(R)$.*

Proof. Since $U \subset R^\times$, $U \cap I = \emptyset$. We know that $(1, 0)$ is an edge in $\Gamma(R, U)$. Since $0 \in I$ and I is homogeneous, $(1, -x)$ is also an edge for each $x \in I$. By definition, this shows that $1 + x \in U$. Let $u \in U$ and $x \in I$, we have $u + x = u(1 + u^{-1}x)$. We know that $1 + u^{-1}x \in U$ and $u \in U$. Hence, $u + x \in U$. The fact that $I \subset \mathrm{Rad}(R)$ follows from [29, Chapter 4.3]). □

We have the following immediate corollary.

Corollary 2.14. *If R is semisimple; namely* $\mathrm{Rad}(R) = 0$*, then there is no two-sided ideal I such that I is homogeneous in* $\Gamma(R, U)$*. In particular, if* $\Gamma(R, U)$ *is connected and anti-connected, then* $\Gamma(R, U)$ *is prime.*

2.2.3. *Unitary Cayley graphs.* In this section, we focus on the case $S = R^{\times}$. In this case, the associated graph is denoted by G_R and is well-known as the unitary Cayley graph of R (see [1, 6]). This type of graph has a rich history; we can trace its root in the work of Evans and Erdős in [10]. In [9, 23, 25], we provide a complete classification of finite commutative ring R such that G_R is connected/prime. In this section, we give a similar answer for all finite ring R.

We note that, by [9, Proposition 4.30], $\mathrm{Rad}(R)$ is a homogeneous set in G_R. By the same argument as in [9, Corollary 4.2], we have the following isomorphism

$$G_R \cong G_{R^{\mathrm{ss}}} * E_n,$$

here $R^{\mathrm{ss}} = R/\mathrm{Rad}(R)$ is the simplification of R, E_n is the empty graph on $n = |\mathrm{Rad}(R)|$ vertices, and $*$ denotes the wreath product of two graphs (see [9, Definition 2.5] for the definition of the wreath product of graphs). This isomorphism shows that G_R is connected if and only if $G_{R^{\mathrm{ss}}}$ is connected. Furthermore, if G_R is prime then $\mathrm{Rad}(R) = 0$; namely $R = R^{\mathrm{ss}}$. Therefore, we can assume from now on that $R = R^{\mathrm{ss}}$. In this case, the Artin-Wedderburn theorem implies that R^{ss} is a product of local semisimple rings

$$R^{\mathrm{ss}} = \prod_{i=1}^{s} R_i \times \prod_{i=1}^{r} M_{d_i}(F_i). \tag{2.2}$$

Here R_i is a finite field such that $2 \leq |R_1| \leq |R_2| \leq \cdots \leq |R_s|$. Additionally, $d_i \geq 2$, and F_i is a finite field. We can then see that $G_{R^{\mathrm{ss}}}$ is a direct product of unitary Cayley graphs

$$G_{R^{\mathrm{ss}}} = \prod_{i=1}^{s} G_{R_i} \times \prod_{i=1}^{r} G_{M_{d_i}(\mathbb{F}_i)} = \prod_{i=1}^{s} K_{|R_i|} \times \prod_{i=1}^{r} G_{M_{d_i}(\mathbb{F}_i)}.$$

We have the following proposition, which is a direct generalization of [9, Lemma 4.33] and [25, Theorem 3.6].

Proposition 2.15. $G_{R^{ss}}$ *(and hence equivalently* G_R*) is connected if and only if in the above decomposition, there is at most one* $i \in \{1, \ldots, s\}$ *such that* $|R_i| = 2$.

Proof. If there is more than two i such that $|R_i| = 2$ then the direct product contains a copy of $K_2 \times K_2$ which is not connected. Therefore, $G_{R^{\mathrm{ss}}}$ is also not connected. Conversely, suppose that there is at most one i such that $|R_i| = 2$. By our ordering, $|R_1| = 2$ and $|R_k| > 2$ for each $2 \leq k \leq s$. For these k, each graph G_{R_k} is connected and non-bipartite. Similarly, for $1 \leq i \leq r$, $G_{M_{d_i}(F_i)}$ is also connected and non-bipartite (see [23, Proposition 3.5]). By [12, Corollary 5.10], $G_{R^{\mathrm{ss}}}$ is connected. □

We now classify R such that G_R is prime. As explained above, if G_R is prime, then R is necessarily semisimple; namely $R = R^{ss}$. Additionally, G_R must also be connected and anti-connected. By Proposition 2.15 if $G_{R^{ss}}$ is connected, there is at most one i such that $|R_i| = 2$ in the Artin-Wedderburn decomposition of R. For anti-connected, we have the following simple observation.

Proposition 2.16. *$G_{R^{ss}}$ is not anti-connected if and only if R^{ss} is a field.*

Proof. If R^{ss} is a field then $G_{R^{ss}}$ is a complete graph. Therefore, its complement is not connected. Conversely, assume that R^{ss} is not a field. There are two cases to consider.

Case 1. R^{ss} is a product of two rings; say $R = R_1 \times R_2$. We claim that for each $(r_1, r_2) \in R_1 \times R_2$, there is a walk in $G^c_{R^{ss}}$ between $(0, 0)$ and (r_1, r_2) (by a translation, this shows that there is a walk between any two vertices in $G_{R^{ss}}$. If r_1, r_2 is not a unit then $(0, 0)$ and (r_1, r_2) are adjacent in $G^c_{R^{ss}}$. Now, suppose that r_1, r_2 are both units. Then have the following walk

$$(0, 0) \to (r_1, 0) \to (r_1, r_2).$$

Case 2. $R^{ss} = M_n(F)$ for some $n > 1$ and F is a field. Let $A = [v_1 \quad v_2 \quad \dots \quad v_n]$ be a matrix formed by column vectors $\{v_1, v_2, \dots, v_n\}$. We claim that there is a walk between 0 and A. In fact, let $A_1 = [v_1 \quad v_2 \quad \dots v_{n-1} \quad 0]$. Then both A_1 and $A - A_1$ are not invertible. Consequently, we have the following walk in $G^c_{R^{ss}}$

$$0 \to A_1 \to A.$$

Since this true for all A, we conclude that $G^c_{R^{ss}}$ is connected. □

We are now ready to state and prove the main theorem about the primeness of $G_{R^{ss}}$.

Proposition 2.17. *G_R is prime if and only if the following conditions are satisfied*

(1) $R = R^{ss}$.

(2) Let

$$R^{ss} = \prod_{i=1}^{s} R_i \times \prod_{i=1}^{r} M_{d_i}(F_i),$$

be the decomposition of R^{ss} into products of fields and matrix rings as in Eq. (2.2). *Then there is at most one* $1 \leq i \leq s$ *such that* $|R_i| = 2$.

(3) R^{ss} is not a field (R^{ss} is a field if and only if $s = 1$ and $t = 0$).

Proof. We have shown that these conditions are necessary in Proposition 2.15 and Proposition 2.16. Let us show that they are sufficient. Condition (1) implies that $G_R = G_{R^{ss}}$. Additionally, condition (2) and (3) imply that G_R is both connected an anti-connected. By Corollary 2.14 and Proposition 2.15, we conclude that G_R is a prime graph. □

2.2.4. *Primeness and singularity of the adjacency matrix.* In [28, Section 4.3], we study a quite interesting relationship between the primeness of a U-unitary graph and the singularity of its adjacency matrix. More precisely, we show that if R is commutative and $U \subset R^{\times}$ such that $\Gamma(R, U)$ is connected, anti-connected, and not prime, then the adjacency matrix of $\Gamma(R, U)$ is singular. Equivalently, 0 is an eigenvalue of $\Gamma(R, U)$(see [28, Proposition 4.16]). This statement holds for any finite ring R. In fact, if $\Gamma(R, U)$ has these properties, then by Proposition 2.11 there exists a two-sided ideal I such that I is a homogeneous set in $\Gamma(R, U)$. Furthermore, since $I \cap U = \emptyset$, $\Gamma(R, U)$ is isomorphic to the wreath product $\Gamma(R/I, U) * E_{|I|}$. By [21, Theorem 3.3], 0 is an eigenvalue of $\Gamma(R, S)$.

We are interested in the converse of this statement.

Question 2.18. If $\Gamma(R, U)$ is prime, is it true that 0 is not an eigenvalue of $\Gamma(R, U)$?

Here, we provide a partial answer to this question for the case of unitary Cayley graphs.

Proposition 2.19. *Suppose that G_R is prime. Then 0 is not an eigenvalue of G_R.*

Proof. By Proposition 2.17, if G_R is prime then R is semisimple; namely

$$R = \prod_{i=1}^{s} R_i \times \prod_{i=1}^{r} M_{d_i}(F_i),$$

where R_i, F_i are fields and $d_i \geq 2$. Therefore

$$G_R \cong \prod_{i=1}^{s} K_{|R_i|} \times \prod_{i=1}^{r} G_{M_{d_i}(F_i)}.$$

Here, all products are the direct product. An eigenvalue of G_R is therefore of the form $\prod_{i=1}^{s} \lambda_i \prod_{i=1}^{r} \lambda_i'$ where λ_i is an eigenvalue of $K_{|G_i|}$ and λ_i' is an eigenvalue of $G_{M_{d_i}(F_i)}$. Since $K_{|G_i|}$ is a complete graph, $\lambda_i \neq 0$. Similarly, by [6, Theorem 1.1], $\lambda_i' \neq 0$ as well. Therefore, all eigenvalues of G_R are not 0. □

3. SPECTRA THEORY OF U-UNITARY CAYLEY GRAPHS OVER A FINITE SYMMETRIC FROBENIUS ALGEBRA

3.1. **Symmetric Frobenius algebras.** Let R be a S-algebra; namely, there is a ring homomorphism $f : S \to R$ such that $f(S)$ is contained in the center of R. To avoid lengthy notations, we will often identify S and its image $f(S)$ in S.

Definition 3.1. We say that R is a symmetric Frobenius S-algebra if there exists a S-module morphism $\psi_{R,S} : R \to S$ satisfying the following conditions

(1) $\psi_{R,S}(r_1 r_2) = \psi_{R,S}(r_2 r_1)$.
(2) The kernel of $\psi_{R,S}$ does not contain any non-zero left ideal in R.

Proposition 3.2. *Suppose that R is a symmetric Frobenius S-algebra, and S is a symmetric Frobenius T-algebra. Then R is a symmetric Frobenius T-algebra.*

Proof. Let $\psi_{R,S} : R \to S$ (respectively $\psi_{S,T} : S \to T$) be the morphism that makes R (respectively S) a symmetric Frobebenius S-algebra (respectively T-algebra). Let $\psi_{R,T}$ be the composition of $\psi_{S,T} \circ \psi_{R,S}$. By definition $\psi_{R,T} : R \to T$ is T-linear and symmetric. We claim that its kernel does not contain any non-trivial left ideal in R. In fact, suppose to the contrary it is not the case. Then, there exists $x \in R$ such that $\psi_{R,T}(rx) = 0$ for all $r \in R$. For each $s \in S$, $sr \in R$ and hence $\psi_{R,T}(srx) = 0$. By definition

$$0 = \psi_{R,T}(srx) = \psi_{S,T}(\psi_{R,S}(srx)) = \psi_{S,T}(s\psi_{R,S}(rx)).$$

Since $\psi_{S,T}$ does not contain a non-trivial left ideal, we conclude that $\psi_{R,S}(rx) = 0$. Since this is true for all $r \in R$, $\ker(\psi_{R,S})$ contains the left ideal generated by x–which is a contradiction. □

Proposition 3.3. *Let R be a finite commutative ring and n a positive integer. Then $M_n(R)$ is a symmetric Frobenius R-algebra under the trace map.*

Proof. Let $T : M_n(R) \to R$ be the trace map. It is known that T is symmetric; namely $T(AB) = T(BA)$ for all $A, B \in M_n(R)$. We now show that T is non-degenerate. In fact, suppose to the contrary that $\ker(T)$ contains a left ideal in $M_n(R)$. Then, there exists a non-zero matrix $A \in M_n(R)$ such that $T(BA) = 0$ for all $B \in M_n(R)$. Let E_{ij} be the matrix whose (i, j)-position is 1 and 0 everywhere else. We have

$$0 = T(E_{ij}A) = A_{ji}.$$

Since this is true for all $1 \leq i, j \leq n$, we must have $A = 0$, which is a contradiction. □

We have the following corollary.

Corollary 3.4. *Let S, R be commutative rings such that R is a Frobenius S-algebra. Then $M_n(R)$ is also a Frobenius S-algebra. In particular, if $\mathbb{F}_q$ is a finite field whose base field is $\mathbb{F}_p$, then $M_n(F)$ is a symmetric Frobenius $\mathbb{F}_p$-algebra.*

Proof. The first part follows from Proposition 3.2 and Proposition 3.3. The second part follows from the first part and the fact that F is a symmetric Frobenius $\mathbb{F}_p$-algebra under the classical trace map $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$. □

3.2. **Spectral description of U-unitary Cayley graphs.** Suppose that R is a symmetric Frobenius $\mathbb{Z}/n$-algebra for some $n > 1$. Let ψ be the associated $\mathbb{Z}/n$-linear functional $\psi : R \to \mathbb{Z}/n$. Let ζ_n be a fixed primtive root of unity in $\mathbb{C}$. Let $\chi : R \to \mathbb{C}$ be the character defined by $\chi(s) = \zeta_n^{\psi(s)}$. For each $r \in R$, let χ_r be the character defined by $\chi_r(s) = \chi(rs)$. The map

$$\Phi : R \to \mathrm{Hom}(R, \mathbb{C}^{\times}),$$

sending $r \mapsto \chi_r$ is group homomorphism (with respect to the additive structure on R). Since ψ is non-degenerate, Φ is injective. However, since R is finite, it is an isomorphism as well. In other words, χ is a generating character for the dual group $\mathrm{Hom}(R, \mathbb{C}^{\times})$ (see [13]). We remark that since ψ is symmetric, for each $r, s \in R$, $\chi_r(s) = \chi_s(r)$.

By the circulant diagonalization theorem (see [16]), we know that the eigenvalues of the Cayley graph $\Gamma(R, S)$ are given by the multiset $\{\lambda_r\}_{r \in R}$ where

$$\lambda_r = \sum_{s \in S} \chi_r(s).$$

Next, we will show that when $\Gamma(R, S)$ is U-unitary, there is a quite elegant description of the above sum by certain supercharacter theory on R. To do so, we first need to recall the definition of a supercharacter theory.

Definition 3.5. (see [28, Definition 2.1]) Let G be a finite abelian group. Let $\mathcal{K} = \{K_1, K_2, \ldots, K_m\}$ be a partition of G and $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ a partition of the dual group $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^{\times})$ of characters of G. We say that $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory for G if the following conditions are satisfied

(1) $\{0\} \in \mathcal{K}$;
(2) $|\mathcal{X}| = |\mathcal{K}|$;
(3) For each $X_i \in \mathcal{X}$, the character sum

$$\sigma_i = \sum_{\chi \in X_i} \chi$$

is constant on each $K \in \mathcal{K}$;
(4) For a fixed $\chi \in \mathcal{X}$ the sum $\sum_{k \in K_i} \chi(k)$ does not depend on the choice of $\chi \in X$.

We now show that each U induces a supercharacter theory on R (this generalizes [28, Theorem 4.1] for the case R is commutative). More precisely

Proposition 3.6. *let $\mathcal{K} = \{K_1, K_2, \ldots, K_m\}$ be the double quotient $U \backslash R / U$. Additionally, let $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ be the partition of the character group of R defined by*

$$X_i = \{\chi_x \mid x \in K_i\}.$$

Then the pair $(\mathcal{K}, \mathcal{X})$ is a symmetric supercharacter theory for R. Furthermore, $(\mathcal{K}, \mathcal{X})$ satisfies Condition 4 in Definition 3.5.

Proof. The first two conditions are clear from the definition of $\mathcal{K}$ and $\mathcal{X}$. Let us prove the third condition. Let

$$\sigma_i = \sum_{x \in K_i} \sigma_x.$$

Suppose that $y \sim_U z$, we will show that $\sigma_i(y) = \sigma_i(z)$. In fact, by definition, $y = u_1 z u_2$ for some $u_1, u_2 \in U$. We then have

$$\begin{aligned} \sigma_i(y) &= \sum_{x \in K_i} \sigma_x(y) = \sum_{x \in K_i} \sigma_x(u_1 z u_2) = \sum_{x \in K_i} \chi(x u_1 z u_2) \\ &= \sum_{x \in K_i} \chi(u_2 x u_1 z) = \sum_{t \in u_1 K_i u_2} \chi_(tz) = \sum_{t \in u_1 K_i u_2} \chi_t(z) = \sum_{x \in K_i} \chi_x(z) = \sigma_i(z). \end{aligned}$$

Here, the third equality follows from the fact that $\chi(rs) = \chi(sr)$. The last equality follows from the fact that $u_1 K_i u_2 = K_i$. We conclude that the pair $(\mathcal{K}, \mathcal{X})$ satifies the third condition of Definition 3.5. Finally, the last condition of Definition 3.5 can be obtained by an almost identical argument as above. $\square$

For each $1 \leq i, j \leq n$ we define, as in [28], the following notation

$$\Omega_{ji} = \sum_{y \in K_j} \chi_{x_i}(k),$$

Here $x_i \in K_i$ (by Proposition 3.6, Ω_{ji} does not depend on the choice of x_i.).

Theorem 3.7. *Let $\Gamma(G,S)$ be an U-unitary Cayley graph. Then, the spectrum of $\Gamma(G,S)$ is the multiset $\{[\lambda_i]_{|K_i|}\}_{1 \leq i \leq m}$ where*

$$\lambda_i = \sum_{K_j \subset S} \Omega_{ji}.$$

Consequently, $\Gamma(G,S)$ has at most m distinct eigenvalues where $m = |U \backslash R / U|$.

Proof. For each $r \in R$, let λ_r be the eigenvalue associated with r under the circulant diagonalization theorem. We then have

$$\lambda_{x_i} = \sum_{s \in S} \chi_{x_i}(s) = \sum_{K_j \subset S} \left(\sum_{kinK_j} \chi_{x_i}(k) \right) = \sum_{K_j \subset S} \Omega_{ji}.$$

We also know that if $x_i \sim_U x_i'$, then $\lambda_{x_i} = \lambda_{x_i'}$. Therefore, each λ_{x_i} occurs $|K_i|$-times. $\square$

3.3. **The case of matrix rings.** In this section, we apply Theorem 3.7 to some U-unitary Cayley graphs over a matrix ring. We will show, in particular, that the upper bound in Theorem 3.7 can be strict in several cases. First, we study the case of gcd-graphs over $M_n(F)$.

Corollary 3.8. *Let $R = M_n(F)$ where F is a finite field. Let $\Gamma(R,S)$ be a $R^\times$-unitary Cayley graph. Then $\Gamma(R,S)$ has at most $(n+1)$ distinct eigenvalues.*

Proof. By Corollary 3.4, we know that $M_n(F)$ is a symmetric Frobenius $\mathbb{F}_p$-algebra where $\mathbb{F}_p$ is the base field of F. By the theory of row and columns operations, for two matrices A and B, $A \sim_{GL_n(F)} B$ if and only if $\text{rank}(A) = \text{rank}(B)$. Therefore, $|Gl_n(F) \backslash M_n(F) / GL_n(F)|$ has exactly $n+1$ equivalence classes. Therefore, by Theorem 3.7, $\Gamma(R,S)$ has at most $n+1$ distinct eigenvalues. $\square$

Remark 3.9. The upper bound in Corollary 3.8 is strict. In fact, in [6, Theorem 1.1], the authors show that the unitary Cayley graph on $M_n(F)$ has exactly $n+1$ distinct eigenvalues. It would be interesting to see whether their calculations could be generalized to all $R^\times$-unitary Cayley graphs over $M_n(F)$.

We now discuss another result which shows the power of Corollary 3.8. Let $R = GL_n(F)$ and $U = SL_n(F)$–the set of all invertible matrices with determinant 1.

Proposition 3.10. $|SL_n(F)\backslash M_n(F)/SL_n(F)| = n + |F| - 1$.

Proof. For $A, B \in M_n(F)$ such that $A \sim_{SL_n(F)} B$, then $\det(A) = \det(B)$. Therefore, if A, B are invertible and $\det(A) \neq \det(B)$, then A and B are not equivalent. This shows that

$$|SL_n(F)\backslash GL_n(F)/SL_n(F)| = |F|^{\times} = |F| - 1.$$

On the other hand, we claim that if $A \sim_{GL_n(F)} B$ and A, B are not invertible, then $A \sim_{SL_n(F)} B$. Let $r = \text{rank}(A) = \text{rank}(B)$. Then $r < n$. Let I_r be the identity matrix of size $r \times r$ and $\widehat{I_r}$ be the following matrix

$$\widehat{I_r} = I_r \oplus 0 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

By row and column operations, we know that there exists $P, Q \in GL_n(F)$ such that $A = P\widehat{I_r}Q$. Let P' (respectively Q') be the new matrix obtained by multiplying the last column of P (respectively Q) by $\frac{1}{\det(P)}$ (respectively $\frac{1}{\det(Q)}$). Then, we still have

$$A = P\widehat{I_r}Q = P'\widehat{I_r}Q'.$$

Furthermore, $\det(P') = \det(Q') = 1$. Therefore, $A \sim_{SL_n(F)} \widehat{I_r}$. Similarly, $B \sim_{SL_n(F)} \widehat{I_r}$, and hence $A \sim_{SL_n(F)} B$. This shows that the rank map gives an isomorphism between $SL_n(F)\backslash[M_n(F) - GL_n(F)]/SL_n(F)$ and the set $\{0, 1, \ldots, n\}$. In summary, we have $|SL_n(F)\backslash M_n(F)/SL_n(F)| = n + |F| - 1$.

□

Corollary 3.11. *Let $\Gamma(M_n(F), S)$ be an $SL_n(F)$-unitary Cayley graph. Then $\Gamma(M_n(F), S)$ has at most $n + |F| - 1$ distinct eigenvalues.*

Remark 3.12. We remark that the upper bound in Corollary 3.11 is also strict in various cases. For example, for $n = 2$, $\mathbb{F}_p$, $R = M_2(\mathbb{F}_p)$ and $U = SL_2(\mathbb{F}_p)$ with $p \in \{5, 7\}$, the number of distinct eigenvalues in $\Gamma(R, U)$ is exactly $p + 1 = n + |F_p| - 1$. However, this upper bound is not strict for $p = 2, 3$.

Let us now study on U-unitary Cayley graphs over the matrix ring $M_n(R)$ where R is a local ring. In general, the theory of matrix ring over R is quite complicated. However, when R is a principal local ideal ring (PIR), there is a quite elegant theory that classifies equivalent classes of matrices over $M_n(R)$. First, we remark that a finite local PIR is necessarily a Frobenius ring (see [13]). By Proposition 3.3, $M_n(R)$ is also a symmetric Frobenius ring. As a result, Theorem 3.7 applies to all U-unitary Cayley graphs over $M_n(R)$. To describe the double coset $GL_n(R)\backslash M_n(R)/M_n(R)$, we first discuss some terminology and notation. Let π be the uniformizer for the maximal ideal of R. Then, there exists a positive integer m such that $\pi^{m-1} \neq 0$ but $\pi^m = 0$. Furthermore, every elements in R can be written in the form $u\pi^a$ where $u \in R^{\times}$ and $0 \leq a \leq m$.

Proposition 3.13. *([4, Theorem 15.24]]) Let R be a local PIR with π as a uniformizer. Then every $A \in M_n(R)$ is $GL_n(R)$-equivalent to a unique diagonal matrix D of the form*

$$D = diag(p^{a_1}, \dots, p^{a_n}),$$

where $0 \leq a_1 \leq a_2 \dots \leq a_n \leq m$ (D is called the Smith normal form of A). Consequently, the number of classes in $GL_n(R)/M_n(R)\backslash GL_n(R)$ is precisely $\binom{m+n}{n}$.

Corollary 3.14. *Let R be a local PIR and $\Gamma(M_n(R), S)$ be a gcd-graph over $M_n(R)$. Then $\Gamma(R, S)$ has at most $\binom{m+n}{n}$ disctint eigenvalues.*

Remark 3.15. The upper bound in Corollary 3.14 could be strict. For example, let $n = 2$ and $R = \mathbb{Z}/4$ (so $m = 2$ in this case). Let S be set of all $A \in M_n(R)$ such that or A is $GL_n(R)$-equivalent to one of the following matrices

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, the gcd-graph $\Gamma(R, S)$ has the following characteritic polynomial

$$(x-81)(x+15)^6(x-17)^9(x-9)^{72}(x+7)^{72}(x+3)^{96}.$$

It has exactly $6 = \binom{m+n}{m} = \binom{4}{2}$ distinct eigenvalues.

4. Perfect state transfer on U-unitary Cayley graphs

Let G be an undirected simple graph with adjacency matrix A_G. Let $F(t)$ be the continuous-time quantum walk associated with G; namely, $F(t) = \exp(\mathrm{i}A_G t)$. We say that there is perfect state transfer (PST) in graph G if there are distinct vertices a and b and a positive real number t such that $|F(t)_{ab}| = 1$. The concept of PST was introduced in [8] in the context of quantum spin networks. Since this pioneering work, there has been a series of articles studying this phenomenon on arithmetic graphs (see [2, 7, 11, 32] for some works in this line of research). One of the main reasons is that a regular graph that has PST must be necessarily integral; namely, all of its eigenvalues are integers (see [11, 32]). Furthermore, we show in [27, Theorem 3.5] that if PST exists, it will only happen at vertices with some strict local conditions.

In [27], we study PST on U-unitary Cayley graphs defined over a finite commutative ring. There, we describe the necessary and sufficient conditions for a U-unitary Cayley graph to have PST (see [27, Theorem 2.2]). In this section, we generalize these results to the non-commutative setting. In particular, we will show that there is no PST on gcd Cayley graphs over $M_n(F)$ where $n > 1$ and F is a finite field.

First, we show that [27, Theorem 2.2] generalizes without much modification to the non-commutative setting (since the argument mostly uses the additive structure of R). Let R be a symmetric Frobenius $\mathbb{Z}/n$-algebra. Let $\psi : R \to \mathbb{Z}/n$ be the non-degenerate functional of R, and let χ be the associated generating character of $\mathrm{Hom}(R, \mathbb{C}^\times)$. Let

$G = \Gamma(R, S)$ be a *generic* Cayley graph over R. The adjacency matrix A_G is an R-circulant matrix (see [16, 5]). For such $r \in R$, let

$$\vec{v}_r = \frac{1}{\sqrt{|R|}}[\chi_r(s)]^T_{s\in R} \in \mathbb{C}^{|R|}.$$

By the circulant diagonalization theorem (see [16]), the set $\{\vec{v}_r : r \in R\}$ forms a normalized orthonormal eigenbasis for all A_G. Furthermore, the eigenvalues of A_G are precisely the multiset $\{\lambda_r\}_{r\in R}$ where $\lambda_r = \sum_{s\in S}\chi_r(s)$. Let $V = [\vec{v}_r]_{r\in R} \in \mathbb{C}^{|R|\times|R|}$ be the matrix formed by this eigenbasis and V^* be the conjugate transpose of V. Then we can write

$$A_G = VDV^* = \sum_{r\in R} \lambda_r \vec{v}_r \vec{v}_r^*,$$

here $D = \operatorname{diag}([\lambda_r]_{r\in R})$ is the diagonal matrix formed by the eigenvalues λ_r. We then have

$$F(t) = \sum_{r\in R} e^{\mathrm{i}\lambda_r t}\vec{v}_r\vec{v}_r^*.$$

Hence

$$F(t)_{s_1,s_2} = \frac{1}{|R|}\sum_{r\in R} e^{\mathrm{i}\lambda_r t}\chi_r(s_1 - s_2) = \frac{1}{|R|}\sum_{r\in R} e^{\mathrm{i}\lambda_r t}\zeta_n^{\psi(r(s_1-s_2))} = \frac{1}{|R|}\sum_{r\in R} e^{2\pi\mathrm{i}\left(\lambda_r\frac{t}{2\pi} + \frac{\psi(r(s_1-s_2))}{n}\right)}.$$

By the triangle inequality, $|F(t)_{s_1,s_2}| = 1$ if and only if $\lambda_r\frac{t}{2\pi} + \frac{\psi(r(s_1-s_2))}{n}$ are constant modulo 1. Furthermore, by symmetry, there exists perfect state transfer between s_1 and s_2 if and only if there exists perfect state transfer between 0 and $s_2 - s_1$. We then have the following criterion, which is a direct generalization of [2, Theorem 4] and [27, Theorem 2.2].

Theorem 4.1. *There exists perfect state transfer from* 0 *to* s *at time* t *if and only if for all* $r_1, r_2 \in R$

$$(\lambda_{r_1} - \lambda_{r_2})\frac{t}{2\pi} + \frac{\psi((r_1 - r_2)s)}{n} \equiv 0 \pmod 1.$$

We will now apply Theorem 4.1 to some classes of U-unitary Cayley graphs. First, we discuss some necessary conditions. Let $\Gamma(R, S)$ be a U-unitary Cayley graph that has PST. Let $\Delta := \Delta_S$ be the abelian group generated by $r_1 - r_2$, where r_1 and r_2 are elements of R such that $\lambda_{r_1} = \lambda_{r_2}$. By Theorem 4.1, we must have $\psi(ds) = 0$ for all $d \in \Delta$. In particular, if $\Delta = R$, then by the non-degeneracy of ψ, s must be 0, and hence PST cannot exist on $\Gamma(R, S)$. This is the case when $R = M_n(F)$.

Proposition 4.2. *Let* $R = M_n(F)$ *where* $n > 1$ *and* F *is a finite field. Let* $U = R^\times = GL_n(F)$. *Suppose that* $\Gamma(R, S)$ *is a* U*-unitary Cayley graph. Then, there is no PST on* $\Gamma(R, S)$.

Proof. By Theorem 3.7, if $u_1, u_2 \in GL_n(F)$, then $\lambda_{u_1} = \lambda_{u_2}$. Consequently, $u_1 - u_2 \in \Delta$. Furthermore, by [19], every matrix in $M_n(F)$ is the difference of two invertible matrices. Therefore, $\Delta = M_n(F)$. This shows that there is no PST on $\Gamma(R, S)$. □

Remark 4.3. The same argument works for a ring of the form $M_n(S)$ where S is a finite Frobenius local commutative ring. In fact, let F be the residue of R. Then, there is a natural ring homomorphism $M_n(S) \to M_n(F)$. Under this map, a matrix $A \in M_n(S)$ is invertible if and only if its image $\bar{A} \in M_n(F)$ is also invertible. As a result, every matrix in $M_n(R)$ is also a sum of two invertible matrices.

We now show that the same statement holds if we take $U = SL_n(F)$.

Proposition 4.4. *Let $R = M_n(F)$ where $n > 1$ and F is a finite field. Let $U = SL_n(F)$. Suppose that $\Gamma(R, S)$ is a U-unitary Cayley graph. Then, there is no PST on $\Gamma(R, S)$.*

Proof. Let V_1 be the set of rank 1 matrices in $M_n(F)$. By the proof of Proposition 3.10, for $A, B \in V_1$, $A \sim_{SL_n(F)} B$. As a result, $\lambda_A = \lambda_B$. Let Δ' be the abelian group generated by $(A - B)$ where $A, B \in V_1$. Then, $\Delta' \subset \Delta$. We claim that $\Delta' = M_n(F)$. In fact, for $v \in \mathbb{F}^n$, we can write $v = v_1 - v_2$ where $v_1, v_2 \neq 0$. Therefore, if X is a matrix with exactly one non-zero column vector, then $X \in \Delta'$. Since every matrix is a sum of those X's, we conclude that $\Delta' = M_n(F)$. Consequently, $\Delta = M_n(F)$ and hence $\Gamma(R, S)$ has no PST. $\square$

Remark 4.5. By Proposition 4.2 and Proposition 4.4, PST cannot exist on U-unitary graphs with $U = GL_n(F)$ or $U = SL_n(F)$. However, PST can exist if U is a smaller subgroup of $GL_n(F)$. For example, let us consider the case $R = M_2(\mathbb{F}_2)$ and U is the subgroup of permutation matrices in R; namely

$$U = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Then $\Gamma(R, U)$ is a dijoint union of four copies of C_4 where C_4 is the cycle graph on 4 vertices. It is known that there is PST on C_4 (see [2, Lemma 9].) It would be interesting to classify all pairs $(M_n(F), U)$ such that there exists a U-unitary graph that has PST.

Acknowledgements

We thank Ján Mináč for his interest in this project and for asking some interesting questions about the upper bound in Theorem 3.7.

References

1. R. Akhtar, M. Boggess, T. Jackson-Henderson, I. Jiménez, R. Karpman, A. Kinzel, and D. Pritikin, *On the unitary Cayley graph of a finite ring*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 117, 13 pages.
2. M. Bašić, M. D. Petković, and D. Stevanović, *Perfect state transfer in integral circulant graphs*, Applied Mathematics Letters **22** (2009), no. 7, 1117–1121.
3. Milan Bašić and Marko D. Petković, *Perfect state transfer in integral circulant graphs of non-square-free order*, Linear Algebra Appl. **433** (2010), no. 1, 149–163. MR 2645074
4. William Clough Brown, *Matrices over commutative rings*, (1993).

5. Sunil K Chebolu, Jonathan L Merzel, Ján Mináč, Lyle Muller, Tung T Nguyen, Federico W Pasini, and Nguyên Duy Tân, *On the joins of group rings*, Journal of Pure and Applied Algebra **227** (2023), no. 9, 107377.
6. Bocong Chen and Jing Huang, *On unitary Cayley graphs of matrix rings*, Discrete Mathematics **345** (2022), no. 1, 112671.
7. Wang-Chi Cheung and Chris Godsil, *Perfect state transfer in cubelike graphs*, Linear Algebra and Its Applications **435** (2011), no. 10, 2468–2474.
8. Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J Landahl, *Perfect state transfer in quantum spin networks*, Physical review letters **92** (2004), no. 18, 187902.
9. Maria Chudnovsky, Michal Cizek, Logan Crew, Ján Mináč, Tung T. Nguyen, Sophie Spirkl, and Nguyên Duy Tân, *On prime Cayley graphs*, arXiv:2401.06062, to appear in Journal of Combinatorics (2024).
10. Paul Erdös and Anthony B. Evans, *Representations of graphs and orthogonal Latin square graphs*, Journal of Graph Theory **13** (1989), no. 5, 593–595.
11. Chris Godsil, *State transfer on graphs*, Discrete Mathematics **312** (2012), no. 1, 129–147.
12. Richard Hammack, Wilfried Imrich, and Sandi Klavžar, *Handbook of product graphs*, CRC press, 2011.
13. Thomas Honold, *Characterization of finite frobenius rings*, Archiv der Mathematik **76** (2001), no. 6, 406–415.
14. Jing Huang, *On the quadratic unitary Cayley graphs*, Linear Algebra and its Applications **644** (2022), 219–233.
15. Gareth A Jones, Ilia Ponomarenko, and Jozef Širáň, *Isomorphisms, Symmetry and Computations in Algebraic Graph Theory: Pilsen, Czech Republic, october 3–7, 2016*, vol. 305, Springer Nature, 2020.
16. Shigeru Kanemitsu and Michel Waldschmidt, *Matrices of finite abelian groups, finite Fourier transform and codes*, Proc. 6th China-Japan Sem. Number Theory, World Sci. London-Singapore-New Jersey (2013), 90–106.
17. Hamide Keshavarzi, Babak Amini, Afshin Amini, and Shahin Rahimi, *Involutory cayley graphs of finite commutative rings*, Journal of Algebra and its Applications (2025).
18. Walter Klotz and Torsten Sander, *Some properties of unitary Cayley graphs*, The Electronic Journal of Combinatorics **14** (2007), no. 1, R45, 12 pages.
19. N. J. Lord, *Matrices as sums of invertible matrices*, Mathematics Magazine (1987).
20. Y. Meemark and S. Sriwongsa, *Perfect state transfer in unitary Cayley graphs over local rings*, Trans. Comb. **3** (2014), no. 4, 43–54. MR 3258993
21. Ján Minác, Lyle Muller, Tung T Nguyen, and Federico W Pasini, *Joins of normal matrices, their spectrum, and applications*, Mathematica Slovaca **75** (2025), 483–498.
22. Ján Minác, Lyle Muller, Tung T Nguyen, and Nguyen Duy Tân, *On the Paley graph of a quadratic character*, Mathematica Slovaca **74** (2024), no. 3, 527–542.
23. Ján Mináč, Tung T Nguyen, and Nguyen Duy Tân, *A complete classification of perfect unitary cayley graphs*, arXiv preprint arXiv:2409.01922 (2024).
24. ______, *On the gcd graphs over polynomial rings*, arXiv preprint arXiv:2409.01929 (2024).
25. Tung T Nguyen and Nguyen Duy Tân, *On gcd-graphs over a finite ring*, preprint (2025).
26. Tung T. Nguyen and Nguyen Duy Tân, *On gcd-graphs over finite rings*, arXiv preprint arXiv:2503.04086 (2025).
27. Tung T. Nguyen and Nguyen Duy Tan, *Perfect state transfer on gcd-graphs*, `https://github.com/tungprime/perfect_state_transfer_on_gcd_graphs`, 2025.
28. Tung T Nguyen and Nguyen Duy Tân, *Supercharacters of finite abelian groups and applications to spectra of u-unitary cayley graphs*, arXiv preprint arXiv:2508.10348 (2025).

29. Richard S. Pierce, *Associative algebras*, Studies in the History of Modern Science, vol. 9, Springer-Verlag, New York-Berlin, 1982. MR 674652
30. Ricardo A. Podestá and Denis E. Videla, *The Waring's problem over finite fields through generalized Paley graphs*, Discrete Mathematics **344** (2021), no. 5, 112324.
31. ———, *Spectral properties of generalized Paley graphs and their associated irreducible cyclic codes*, Australasian Journal of Combinatorics **91** (2025), no. 3, 326–365.
32. Nitin Saxena, Simone Severini, and Igor E Shparlinski, *Parameters of integral circulant graphs and periodic quantum dynamics*, International Journal of Quantum Information **5** (2007), no. 03, 417–430.
33. Wasin So, *Integral circulant graphs*, Discrete Mathematics **306** (2006), no. 1, 153–158.
34. Borworn Suntornpoch and Yotsanan Meemark, *Cayley graphs over a finite chain ring and gcd-graphs*, Bulletin of the Australian Mathematical Society **93** (2016), no. 3, 353–363.
35. Issaraporn Thongsomnuk and Yotsanan Meemark, *Perfect state transfer in unitary Cayley graphs and gcd-graphs*, Linear Multilinear Algebra **67** (2019), no. 1, 39–50. MR 3885879

DEPARTMENT OF MATHEMATICS, ELMHURST UNIVERSITY, ILLINOIS, USA
Email address: `tung.nguyen@elmhurst.edu`

FACULTY MATHEMATICS AND INFORMATICS, HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY, 1 DAI CO VIET ROAD, HANOI, VIETNAM
Email address: `tan.nguyenduy@hust.edu.vn`